

Mach's principle: Exact frame-dragging via gravitomagnetism in perturbed Friedmann-Robertson-Walker universes with $K = (\pm 1, 0)$

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We show that there is *exact dragging* of the axis directions of local *inertial frames* by a weighted average of the cosmological energy currents via *gravitomagnetism* for *all* linear perturbations of *all* Friedmann-Robertson-Walker (FRW) universes and of Einstein's static closed universe, and for *all* energy-momentum-stress tensors and in the presence of a cosmological constant. This includes FRW universes arbitrarily close to the Milne Universe and the de Sitter universe. Hence the *postulate formulated by Ernst Mach* about the *physical cause* for the time-evolution of inertial axes is shown to hold in General Relativity for linear perturbations of FRW universes. — The time-evolution of local inertial axes (relative to given local fiducial axes) is given experimentally by the precession angular velocity $\vec{\Omega}_{\text{gyro}}$ of local gyroscopes, which in turn gives the operational definition of the gravitomagnetic field: $\vec{B}_g \equiv -2\vec{\Omega}_{\text{gyro}}$. The gravitomagnetic field is caused by energy currents \vec{J}_ϵ via the *momentum constraint*, Einstein's $G_i^{\hat{0}}$ equation, $(-\Delta + \mu^2)\vec{A}_g = -16\pi G_N \vec{J}_\epsilon$ with $\vec{B}_g = \text{curl } \vec{A}_g$. This equation is analogous to Ampère's law, but it holds for all *time-dependent* situations. Δ is the *de Rham-Hodge Laplacian*, and $\Delta = -\text{curl curl}$ for the vorticity sector in Riemannian 3-space. — In the solution for an open universe the $1/r^2$ -force of Ampère is replaced by a *Yukawa force* $Y_\mu(r) = (-d/dr)[(1/R)\exp(-\mu r)]$, *form-identical* for FRW backgrounds with $K = (-1, 0)$. Here r is the measured geodesic distance from the gyroscope to the cosmological source, and $2\pi R$ is the measured circumference of the sphere centered at the gyroscope and going through the source point. The scale of the exponential cutoff is the H -dot radius, where H is the Hubble rate, dot is the derivative with respect to cosmic time, and $\mu^2 = -4(dH/dt)$. Analogous results hold in closed FRW universes and in Einstein's closed static universe. — We list six fundamental tests for the principle formulated by Mach: all of them are explicitly fulfilled by our solutions. — We show that only energy currents in the toroidal vorticity sector with $\ell = 1$ can affect the precession of gyroscopes. We show that the harmonic decomposition of toroidal vorticity fields in terms of vector spherical harmonics $\vec{X}_{\ell m}^-$ has radial functions which are *form-identical* for the 3-sphere, the hyperbolic 3-space, and Euclidean 3-space, and are form-identical with the spherical Bessel-, Neumann-, and Hankel functions. — The Appendix gives the de Rham-Hodge Laplacian on vorticity fields in Riemannian 3-spaces by equations connecting the calculus of differential forms with the curl notation. We also give the derivation of the Weitzenböck formula for the difference between the de Rham-Hodge Laplacian Δ and the “rough” Laplacian ∇^2 on vector fields.

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I. SUMMARY AND CONCLUSIONS

A. Conclusions

The time-evolution of *local inertial axes*, i.e. the local *non-rotating frame*, is experimentally determined by the spin axes of *gyroscopes*, e.g. by inertial guidance systems. This has been first demonstrated by Leon Foucault in 1852.

It is an *observational fact* that the spin axes of gyroscopes far away from Earth do not precess relative to quasars. This observational fact has been named “*Mach zero*” by Bondi and Samuel [1]. — Near Earth there are two corrections predicted by General Relativity: (1) The de Sitter precession, which is due to the gyroscope's motion in the curved Schwarzschild metric, and which has been measured. (2) The extremely small frame-dragging

effect by the rotating Earth, the Lense-Thirring effect, which is predicted to be 43 milli-arc-sec per year, which hopefully will be extracted from the data taken by Gravity Probe B [2], and whose measurement by the LAGEOS satellites has been reported in [3]. — The dragging effect by the rotating Earth (and also the perihelion shift of Mercury) is *measured relative to distant quasars*. Therefore a measurement by Gravity Probe B and the LAGEOS satellites (and a measurement of the perihelion shift) is a *test of two things combined*, on the one hand a test of Einstein's General Relativity, on the other hand a test of the observational fact “Mach zero”.

But the fundamental question remains: *What physical cause* explains the observed time-evolution of gyroscope axes, the observational fact “Mach zero”? In J.A. Wheeler's words: Who gives the marching orders to the spin axes of gyroscopes, i.e. to inertial axes? Can theory predict resp. explain the extremely precise observational tests of “Mach zero”?

An answer to this fundamental question was *formulated by Ernst Mach* [4, 5] in his *postulate* that inertial

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axes *exactly follow* an average of the motion of masses in the Universe. Mach's postulate demands *exact frame dragging*, not merely a little bit of frame dragging as in the Lense-Thirring effect. — Mach did not have a mechanism for dragging. The new force came with General Relativity: Gravitomagnetism, exemplified in the Lense-Thirring effect. — Mach also did not know, what average of the cosmological mass currents should be taken.

Mach's starting postulate was: *No absolute space*, in modern parlance *no absolute element* in the input. Only relative motion of physical objects is significant. — From this requirement follows Mach's postulate of *frame invariance* of observable predictions under time-dependent rotations of the reference frame. In Mach's words [5]:

“Obviously it does not matter, whether we think of the Earth rotating around its axis, or if we imagine a static Earth with the celestial bodies rotating around it.”

From these requirements follows Mach's postulate [4]: Centrifugal forces and the rotation of the plane of Foucault's pendulum are *totally determined* by the relative motion of masses in the universe with respect to Earth-fixed axes. In modern parlance: The time-evolution of local inertial axes is totally determined by the energy currents in the universe including the effect of gravitational waves. — The demands “no absolute element,” “frame invariance under rotational motion,” and “totally determined” necessitate the postulate of *exact dragging* of inertial axes by energy currents in the universe. With only partial dragging it would be impossible to fulfill any of the three postulates “no absolute element in the input”, “frame invariance of the solutions”, and “time evolution of inertial axes totally determined by cosmological energy currents”.

For a purely *local analysis*, if one has only laboratory experiments available (like Newton's bucket experiment), the local nonrotating frame is an absolute element. — But, in contrast, for a *global analysis*, if one considers observations of the entire universe, the local nonrotating frame is not an absolute element in the input, because (according to Mach) the nonrotating frame is totally determined by mass motions (energy currents) in the universe, it is nonrotating *relative* to an average of the energy currents in the universe.

The “*relativists*” (Huyghens, Berkeley, Leibniz, Mach, and others) wanted a theory without absolute space (without an absolute element in the input), a theory, where only relative motion is significant. After Einstein's theory of General Relativity it was soon recognized that this theory, when applied to the solar system, is not a “Theory of Relativity” in the sense that the solutions violate the postulates “no absolute element”, “solutions frame invariant”, “totally determined”, and “exact dragging”, see Sec. IH. This is why Einstein and others started to focus on cosmological models. The unperturbed Friedmann-Robertson-Walker models satisfy Mach's postulates. In this paper we derive the new result

that all FRW models with arbitrary linear perturbations also satisfy Mach's postulates. This means that *Cosmological General Relativity* (at the level of linear perturbations of FRW universes) is a true “*Theory of Relativity*”.

We have shown in [6, 7] that the axis directions of local *inertial frames* everywhere in the universe *exactly follow* a weighted average of the energy currents in the universe, i.e. there is *exact frame-dragging*, for linear perturbations of a Friedmann-Robertson-Walker (FRW) background with $K = 0$ (i.e. spatially flat). The *weight function* has an exponential cutoff at the H -dot radius, where H is the Hubble rate, and dot is the derivative with respect to measured cosmic time. Hence we have demonstrated the validity of the hypothesis formulated by Mach about the *physical cause* for the time-evolution of inertial-frame axes for linear perturbations of a FRW background with $K = 0$. The proof follows from Einstein's G_{0i} equation, the *momentum constraint*. The dynamical mechanism is *cosmological gravitomagnetism*.

In the present paper we show that results analogous to those derived for $K = 0$ in [7] are valid for linear perturbations of FRW backgrounds with $K = \pm 1$, i.e. backgrounds which are spatially spherical (S^3) resp. hyperbolic (H^3).

Many alternative and inequivalent postulates have been proposed under the heading “Mach's principle” by many authors after E. Mach. An incomplete list of 10 inequivalent versions of Mach's principle has been given by Bondi and Samuel [1] (based in part on [8]). While the versions “Mach 9” and “Mach 10” listed by [1] are closely related to our “Fundamental Tests 3 and 4” in Sec. VH, their versions “Mach 1-8” definitely do not correspond to anything in the writings of Mach. Furthermore each one of their versions “Mach 1-8” is either not true in cosmological General Relativity, as stated already in [1], or irrelevant, as we explain in Sec. IJ.

A more detailed introduction to the conceptual issues of the postulate formulated by Mach and discussions of publications [9]-[16] (mostly cosmological as postulated by Mach) are given in Secs. I and II of the companion paper [7]. Additional references are given in [8, 12, 13, 17, 18]. Comparisons of our methods and results with the recent papers by Bičák, Lynden-Bell, and Katz [16, 19] are at the end of Sects. IF and VH. — We use the conventions of Misner, Thorne, and Wheeler [20].

B. Vorticity perturbations

We now summarize the main results of the present paper for the case when only vorticity perturbations (= 3-vector perturbations, by definition divergenceless) are present.

Two important results from cosmological perturbation theory [21] for $K = (0, \pm 1)$ are needed to understand the following summary. In the *vorticity sector*:

1. The *slicing* of space-time in slices Σ_t of fixed time, i.e. 3-spaces, is *unique*. The *lapse* function (elapsed

measured time between slices) and g_{00} are *unperturbed*.

2. The *intrinsic geometry* of each slice Σ_t , i.e. of 3-space, remains *unperturbed*.

Details and proofs are given in Sec. III of [7]. Note that Secs. III - V of [7] are written for FRW backgrounds with $K = (0, \pm 1)$.

For the 3-geometry, which remains unperturbed (E^3 , resp. S^3 , resp. H^3), and for our fixed-time problem (the momentum constraint), the uniquely appropriate coordinate choice (gauge choice) is comoving spherical coordinates, $x^i = (\chi, \theta, \phi)$, where $(a\chi)$ is the measured radial distance with $a(t) =$ scale factor. With this gauge choice the coefficients of the 3-metric $^{(3)}g_{ij}$ are *also unperturbed*. The spatial coordinate system is *geometrically rigid* apart from the uniform Hubble expansion. Also g_{00} is unperturbed, and we choose the time-coordinate t to be the measured cosmic time. Hence, with our gauge choice the *only* quantity referring to vorticity perturbations is the *shift* 3-vector β^i (resp $\beta_i = g_{0i}$),

$$ds^2 = -dt^2 + a^2[d\chi^2 + R_{\text{com}}^2(d\theta^2 + \sin^2\theta d\phi^2) + 2\beta_i dx^i dt], \quad (1)$$

where R_{com} stands for R_{comoving} , and $R_{\text{com}}(\chi) = (\chi, \sin\chi, \sinh\chi)$ for $K = (0, +1, -1)$. The index of the shift 3-vector is lowered and raised by the 3-metric, e.g. $\beta_i = g_{ij}\beta^j$.

Our choice of unperturbed comoving spherical coordinates for (E^3, S^3, H^3) is *singled out* by the fact that our radial coordinate lines (θ and ϕ fixed) are *geodesics* in 3-space, an important property in discussions of Mach's principle. In all other gauges, where perturbed 3-metrics $g_{\mu\nu}$ are used, the radial coordinate lines wind up more and more into "spirals" (as time goes on) relative to geodesics on Σ_t . Hence all other gauges (particularly time-orthogonal) give a very awkward way to coordinate unperturbed (E^3, S^3, H^3) spaces.

Up to now the coordinatization (gauge) is not yet completely fixed, there still exists the freedom of *residual gauge transformations* by time-dependent, spatially *rigid rotations* of the coordinates for 3-space. Rigid rotations are crucial in Mach's statement on frame-invariance [5]: See the quote in Sec. IA, "Obviously it does not matter ...".

We shall show in Sec. IH that our solution of the equations of cosmological gravitomagnetism is explicitly *form-invariant* under time-dependent, spatially rigid rotations of the reference frame, i.e. *frame-independent* (valid in all reference frames connected by time-dependent rigid rotations). In contrast, the *solutions* of the equations of General Relativity for the *solar system* are *not* form-invariant under spatially rigid, time-dependent rotations.

Measuring the angular velocity of a given celestial body requires that one first fixes the residual gauge freedom, i.e. one must first state, relative to what one wants to measure the angular velocity. This can be done by making a definite choice of spatial axis directions along one

world line. In the spirit of Mach one could fix the axis directions either along the world-line of the Earth to Earth-fixed axes, or (in an open, asymptotically unperturbed universe) to asymptotic quasars.

C. Gravitomagnetic and gravitoelectric fields

The *general operational definitions* (valid beyond perturbation theory) of the gravitomagnetic field \vec{B}_g and the gravitoelectric field \vec{E}_g are given via measurements by *fiducial observers* (FIDOs) with nonintersecting world lines and with their local ortho-normal bases, LONBs [22]. These FIDOs measure the first time-derivatives of the LONB components of the 3-momentum, p_i , for free-falling quasistatic test particles, resp. of the spin-vector, S_i , for comoving gyroscopes,

$$\text{free-falling quasistatic test particle: } \frac{d}{dt}(p_i) \equiv mE_i^g, \quad (2)$$

$$\text{gyro comoving with FIDO: } \frac{d}{dt}(S_i) \equiv -\frac{1}{2}(\vec{B}_g \times \vec{S})_i, \quad (3)$$

$$\Omega_i^{\text{gyro}} = -\frac{1}{2}B_i^g, \quad (4)$$

where Ω_i are the LONB components of the precession rate of the spin axes of a gyroscope. Hats over indices denote components in a local ortho-normal basis (LONB), and t is the local time measured by the FIDO. Note that LONB components, e.g. p_i , are *directly measurable* (for a given field of LONBs), while coordinate-basis components, e.g. p_μ , are *not measurable without prior knowledge of the metric $g_{\mu\nu}$* .

The operational definitions of \vec{B}_g and \vec{E}_g are *identical* with the operational definitions of the ordinary electric and magnetic fields via measurements on charged point test-particles and on charged spinning test particles, except that the charge q is replaced by the mass m for quasistatic test particles.

The (\vec{B}_g, \vec{E}_g) -fields depend on the choice of FIDOs. For free-falling FIDOs $\vec{E}_g \equiv 0$, and for nonrotating FIDOs (relative to axes of local gyroscopes) $\vec{B}_g \equiv 0$, which are the twin conditions in the equivalence principle. — The (\vec{E}_g, \vec{B}_g) fields are directly related to connection coefficients: $(\omega_{i\hat{0}})_{\hat{0}} \equiv -E_i^g$, $(\omega_{i\hat{j}})_{\hat{0}} \equiv -\frac{1}{2}B_{ij}^g$, where $B_{ij} \equiv \varepsilon_{ijk}B_k$, $\Omega_{ij} \equiv \varepsilon_{ijk}\Omega_k$, and $\Omega_{ij} = (\omega_{ij})_{\hat{0}}$. More details about the general operational definitions of (\vec{E}_g, \vec{B}_g) are given in Sec. IV of [7].

Our *specific choice* for a field of *FIDOs*: Our FIDOs are at fixed values of our coordinates x^i , and we fix the spatial axis directions of our FIDOs in the directions of our coordinate basis vectors $(\partial/\partial\chi, \partial/\partial\theta, \partial/\partial\phi)$. — We recall from Sec. IB that our choice of unperturbed comoving spherical coordinates for (E^3, S^3, H^3) is *singled out* by the fact that the radial coordinate lines (θ and ϕ fixed) are *geodesics* in 3-space. Additional details about

our specific choice of FIDO's are given in [7] at the end of Sec. IV.

The *gravitomagnetic vector potential* \vec{A}_g in the vorticity sector is uniquely determined by $\vec{B}_g \equiv \text{curl } \vec{A}_g$, because $\text{div } \vec{A}_g \equiv 0$. — With $\beta_i \equiv g_{0i}$ and A_i^g defined independently above, it follows that the *shift vector* $\vec{\beta}$ is equal to the *gravitomagnetic vector potential* \vec{A}_g , see also Sec. V of [7]. — Our gravitomagnetic vector potential \vec{A}_g is directly proportional to *Bardeen's gauge-invariant amplitude* Ψ as shown in Eq. (13).

In Sec. III we shall consider gravitomagnetism on a Minkowski background and show that our FIDO's (and hence our \vec{E}_g and \vec{B}_g fields) are *singled out* by the property that all 13 nonzero components of the *Riemann tensor* are *uniquely determined* by spatial and time derivatives of our \vec{B}_g field alone. Our \vec{B}_g field gives an efficient and physically transparent way to represent all information in the Riemann tensor. — In the opposite direction: We shall show that our \vec{B}_g field is determined by the Ricci components $R_{\hat{k}\hat{0}}$ up to a homogeneous \vec{B}_g field, which is equivalent to a time-dependent rigid rotation of the coordinates and FIDO axes according to Eq. (4).

D. The momentum constraint

The crucial equation for Mach's principle is the momentum constraint, Einstein's G_i^0 equation, for cosmological vorticity perturbations. The momentum constraint for perturbations of *all three* FRW background geometries, $K = (\pm 1, 0)$, is given by *one form-identical equation without curvature terms*:

$$(-\Delta + \mu^2) \vec{A}_g = -16\pi G_N \vec{J}_\varepsilon, \quad (5)$$

where $(\mu/2)^2 \equiv -(dH/dt) \equiv (\text{Hubble-dot radius})^{-2}$. Derivation of Eq. (5) in Sec. II. Our momentum constraint has the form of the *inhomogeneous Helmholtz equation* with the sign of the μ^2 -term opposite to the usual sign. The sign in Eq. (5) causes an exponential cutoff, which becomes important beyond the Hubble-dot radius. — The source \vec{J}_ε with $J_i^\varepsilon \equiv T_i^0$ for all types of matter is the directly *measurable* energy-current density, which is equal to the measurable momentum density (for $c = 1$), hence the name “momentum constraint”. “Directly measurable” means that J_i^ε is measurable *without prior knowledge* of g_{0i} , which is the output of solving the momentum constraint. — The energy current \vec{J}_ε is analogous to the charge current \vec{J}_q in Ampère's law. For perfect fluids and linear perturbations $\vec{J}_\varepsilon = (\rho + p)\vec{v}$. — All symbols in Eq. (5) refer to the physical scale, they are not rescaled to a comoving length scale. Therefore the scale factor $a(t)$ does *not* appear in Eq. (5).

It is remarkable that Eq. (5) holds for *time-dependent* gravitomagnetodynamics. All the same, it is an *elliptic* partial differential equation: *No partial time-derivatives of the perturbation field* \vec{A}_g . Time derivatives only appear

in the given background, $H = a^{-1}da/dt$, $\dot{H} = dH/dt$. — Eq. (5) has the same form as Ampère's law for stationary magnetism except for the $\mu^2 \vec{A}_g$ term. — The analogue of the Maxwell term $(\partial_t \vec{E})$ in the Ampère-Maxwell equation is *absent* for the *time-dependent* context of gravitomagnetism. No inconsistency arises when taking the divergence of Eq. (5), since $\text{div } \vec{A}_g \equiv 0$ and $\text{div } \vec{J}_\varepsilon \equiv 0$ in the vorticity sector. The analogue of a Maxwell term cannot be present in gravitomagnetism, this would produce gravitational vector waves, which is impossible. Additional explanations about the momentum constraint are given after Eq. (36) in Sec. VI of [7].

Other perturbed Einstein equations and the perturbed evolution equations for matter will not be needed in the context of this paper. The full set of field equations for linear cosmological gravitomagnetism is given in Secs. V and VI of [7].

E. The de Rham-Hodge Laplacian on vector fields in Riemannian 3-spaces

The *absence* of a *3-curvature term* in Eq. (5) is connected to our use of the *de Rham-Hodge Laplace operator* Δ , which is called “the” Laplace operator in the literature on differential forms in Riemannian geometry [23]–[26]. Except when acting on scalar fields, the de Rham-Hodge Laplace operator Δ must be distinguished from $\nabla^2 = g^{\mu\nu} \nabla_\mu \nabla_\nu$, named the “rough Laplacian” in [24, 25]. — Unfortunately ∇^2 has been used in all publications on cosmological vector perturbations (as far as we know), e.g. in [21, 27], and moreover ∇^2 has been called Laplacian and/or denoted by Δ in many publications on vector perturbations, e.g. in [28]–[31]. Using ∇^2 has the unfortunate consequence that curvature terms appear, where none are present when the de Rham-Hodge Laplacian Δ is used. The most important example is electromagnetism in curved space-time, where the equivalence principle forbids curvature terms, if the de Rham-Hodge Laplacian resp d'Alembertian is used [32].

In Riemannian 3-space, the de Rham-Hodge Laplacian acting on vorticity fields, i.e. fields with $\text{div } \vec{a} = \star d \star \vec{a} = d\vec{a} = 0$, is given by

$$(\Delta \vec{a})_\mu = -(\text{curl curl } \vec{a})_\mu = -(\star d \star d \vec{a})_\mu, \quad (6)$$

where we have given both the notation of elementary vector calculus and the notation of differential forms with $d \equiv$ exterior derivative, $(-\delta) \equiv d^* \equiv$ adjoint operator to the exterior derivative, $\star \equiv$ Hodge dual, tilde denoting p -forms, and $(\vec{a})_\mu \equiv (\tilde{a})_\mu$. Note that $(\text{curl } \vec{a})_\mu = (\star d \vec{a})_\mu$. Details are given in the Appendix.

The de Rham-Hodge Laplacian is *singled out* among the elliptic differential operators by the following properties: (1) If in some region all types of sources are zero, $\text{div } \vec{v} = 0$ and $\text{curl } \vec{v} = 0$, then also $\Delta \vec{v} = 0$. (2) Δ commutes with d, δ and \star . Therefore applying curl, div, and grad to an eigenfield of the Laplacian produces

again an eigenfield of the Laplacian with the same eigenvalue. — The “rough Laplacian” ∇^2 does not have properties (1) and (2), if the Ricci tensor is not zero. — The de Rham-Hodge Laplacian is also easier to compute than the “rough Laplacian”, because no Christoffel symbols resp. connection coefficients are needed to compute it.

The difference between the de Rham-Hodge Laplacian and the “rough Laplacian” is given by the *Weitzenböck* formula, i.e. by curvature terms. The Weitzenböck formula for vector fields, is derived in Sec. 3 of the Appendix. For a FRW universe with $K = \pm 1$ and with the curvature radius $a_c \equiv 1$, the Weitzenböck formula gives $\Delta \vec{A} = (\nabla^2 - 2K)\vec{A}$.

The de Rham-Hodge Laplacian applies to totally antisymmetric tensor fields. For *totally symmetric* tensor fields, e.g. for cosmological tensor perturbations, the appropriate Laplacian is the *Lichnerowicz Laplacian* [33, 34]. For vector fields the Laplacians of Lichnerowicz and of de Rham-Hodge coincide.

F. No absolute element in the input: The exponential cutoff at the H -dot radius

The *crucial difference* between our momentum constraint, Eq. (5), and the corresponding equation of Bardeen, Eq. (4.12), and the equation advocated by Bičák et al [16, 19] is our $\mu^2 \vec{A}_g$ term, which causes the *Yukawa cutoff* in Eq. (7) below, and which is absent in the solution of the equation advocated by Bičák et al. Our $\mu^2 \vec{A}_g$ term is due to the difference of the *sources*.

Our guiding principle and starting point: “*No Absolute Element in the Input*”. The source, the matter *input*, must be *measurable without prior knowledge* of $g_{0i} = A_i^{(g)}$. The latter is the *output* of solving the momentum constraint. From Eq. (4) we see that $\vec{B}_g = \text{curl} \vec{A}_g$ directly gives the local nonrotating frame all over the universe, which should not be in the input, because it would be an absolute element in the input. — We write the momentum constraint in terms of the *LONB components* T_i^0 , which are measurable without prior knowledge of \vec{A}_g , i.e. without an absolute element in the input.

For Mach’s principle, the relevant factor in the source is the *angular velocity* of cosmic matter, $\vec{\Omega}_{\text{matter}}$, measured from our position and *relative to geodesics* (on Σ_t) which start in the directions of our chosen fiducial axes. In the *local ortho-normal bases*, the transverse part of the energy current $T_i^0 = J_i^{(\varepsilon)}$ is directly proportional to Ω_i^{matter} (relative to our fiducial axes), which can be measured *without prior knowledge* of $g_{0i} = A_i^{(g)}$. The measured transverse matter velocity is directly proportional to Bardeen’s *gauge-invariant* velocity amplitude v_s , which he does *not* use in his momentum constraint.

In contrast, Bardeen’s Eq. (4.12) (and the equation advocated by Bičák et al) is written in the *coordinate basis* with the source T_i^0 , and this is proportional to Bardeen’s

other gauge-invariant velocity amplitude v_c . The curl of the velocity field with the amplitude v_c is proportional to the angular velocity of matter *relative to the axes of gyroscopes* at the positions of the cosmic sources. But this has the severe drawback that v_c is *not measurable without prior knowledge* of the time-evolution of gyroscope axes all over the cosmos, which should be the *output* of solving the momentum constraint: $\vec{\Omega}_{\text{gyro}} = -\frac{1}{2}\vec{B}_g = -\frac{1}{2}\text{curl}\vec{A}_g$ with $A_i^{(g)} = g_{0i}$. Bardeen’s version of the momentum constraint (and the version advocated by Bičák et al) needs an *absolute element in the input*: the local nonrotating frames all over the cosmos. This violates the crucial starting demand: No absolute element in the input. The requirement “the theory should not contain an absolute element”, discussed in [35], is closely connected to Mach’s starting point: *No absolute space*.

Einstein commented about the energy-momentum tensor $T_{\mu\nu}$ in a *coordinate basis*: “If you have a tensor $T_{\mu\nu}$ and not a metric ... the statement that matter by itself determines the metric is meaningless” [36]. This objection of Einstein directly applies to the approach advocated by Bičák et al [16, 19]. On the other hand we have shown in Sec. IX of [7] that this objection of Einstein does not apply to the *LONB components* of the energy-momentum tensor.

G. The solution of the momentum constraint

The solution of the momentum constraint Eq. (5) for cosmological gravitomagnetism on a background of *open* FRW universes is given by *identical expressions* for $K = 0$ and $K = -1$,

$$\vec{B}_g(P) = -4G_N \int d(\text{vol}_Q) [\vec{n}_{PQ} \times \vec{J}_\varepsilon(Q)] Y_\mu(r_{PQ}), \quad (7)$$

$$Y_\mu(r) = \frac{-d}{dr} \left[\frac{1}{R} \exp(-\mu r) \right] = \text{Yukawa force}, \quad (8)$$

where r_{PQ} is the measured geodesic (radial) distance from the field point P (gyroscope) to the source point Q , and $(2\pi R_{PQ})$ is the measured circumference of the great circle through Q with the center P . Hence $r = a\chi$, where a is the scale factor, and χ is the radial coordinate (co-moving geodesic distance from the origin). — The structure of Eq. (7) is analogous to *Ampère’s force* for ordinary stationary magnetism, however the $1/r^2$ of Ampère’s force is replaced by the *Yukawa force* of Eq. (8). — In the curved space H^3 the geodesic distance r_{PQ} appears in the exponential cutoff and in the derivative of Eq. (8), while R_{PQ} appears in the denominator.

Although the structure of Eq. (7) is analogous to the solution of Ampère’s law for ordinary stationary magnetism, there is a *fundamental difference* to Ampère’s law, since Eq. (7) is valid for *time-dependent* situations, i.e. gravitomagneto-dynamics.

The *vector structure* of Eq. (7) needs explanation, since there is no global parallelism on a hyperbolic 3-space H^3 .

The vectors $\vec{J}_\varepsilon(Q)$ and \vec{n}_{PQ} are in the tangent space at the source point Q , and \vec{n}_{PQ} is the unit vector pointing along the geodesic from P to Q . We parallel transport $[\vec{n}_{PQ} \times \vec{J}_\varepsilon(Q)]$ along the geodesic from Q to P , where the contributions from all sources are added. — We use the arrow-notation of Thorne et al [22] for vectors in tangent spaces to Riemannian 3-spaces from a $(3+1)$ -split. This emphasizes the structure analogous to Ampère’s law.

H. Frame invariance, no boundary conditions

There is a second *fundamental difference* between the solution Eq. (7) for cosmological gravitomagnetism and the corresponding solutions in other theories, Ampère’s magnetism, electromagnetism in Minkowski space, and general relativity for the solar system. The *solutions* in all these *other* theories contain an *absolute element*: Ampère’s law and the laws of electromagnetism in Minkowski space do not hold in frames which are rotating relative to inertial frames, unless one introduces fictitious forces.

Einstein’s *equations* by themselves are form-invariant. But for general relativity of the solar system, when working in rotating frames relative to the asymptotic nonrotating frame (the absolute element), one must impose *boundary conditions* on the solutions in the asymptotic Minkowski space. The boundary conditions *encode the absolute element*, they explicitly encode fictitious forces (which do not have sources within the solar system), as explained in Sec. XI of [7]. Hence the *solution* of Einstein’s equations for the solar system is *not form-invariant* when going to a frame which is rotating relative to asymptotic inertial frames.

In contrast, the solution for *cosmological* General Relativity, Eq. (7), remains valid as it stands, i.e. the *solution is form-invariant*, if one goes to a frame which is in globally rigid rotation relative to the previous reference frame: The form of the solution is *frame-independent*, because both sides of Eq. (7) change by the same term. Therefore the solution Eq. (7) contains *no absolute element*. — *No boundary conditions* at spatial infinity are needed for regular solutions in Eq. (7).

I. Exact dragging of inertial axes

The solution Eq. (7) can be rewritten to show exact dragging of inertial axes explicitly. For reasons of *symmetry* under rotations and space reflection relative to a gyroscope at the origin, a general velocity field of matter for r_{PQ} fixed can contribute to the gyroscope’s precession only through its term with $(\ell = 1, P = +)$ in the spherical harmonic decomposition. This term is a toroidal vorticity field, and it is equivalent to a rigid rotation with the angular velocity $\vec{\Omega}_{\text{matter}}^{\text{equiv}}(r)$, see Sec. V A.

From Eqs. (4, 7) we obtain

$$\vec{\Omega}_{\text{gyro}} = \langle \vec{\Omega}_{\text{matter}}^{\text{equiv}} \rangle \equiv \int_0^\infty dr \vec{\Omega}_{\text{matter}}^{\text{equiv}}(r) W(r), \quad (9)$$

$$W(r) = \frac{1}{3} 16\pi G_N(\rho + p) R^3 Y_\mu(r) \quad (10)$$

for perturbations of open FRW universes. $Y_\mu(r)$ is the Yukawa force given in Eq. (8).

The crucial equation for establishing *exact* (as opposed to partial) dragging of local inertial axis directions is the condition that the *weight function* $W(r)$ must be *normalized to unity*,

$$\int_0^\infty dr W(r) = 1, \quad (11)$$

as it must be for a *proper averaging weight function* in *any* problem. Our weight function with its Yukawa force, Eq. (10), fulfills Eq. (11) as shown in Sec. V C.

Eqs. (9, 11) state our most important result: $\vec{\Omega}_{\text{gyro}}$ is equal to the weighted average (with proper normalization to unity) of $\vec{\Omega}_{\text{matter}}^{\text{equiv}}(r)$ for all types of matter-energy and for all types of energy-current distributions. The time-evolution of inertial axes *exactly follows* the weighted average of cosmic matter motion. Hence the evolution of inertial axes is *fully determined* (not merely influenced) by the cosmic energy currents. There is *exact dragging* (not merely partial dragging) of inertial axes by the *weighted average* of cosmic energy currents, as postulated by Ernst Mach [4, 5]. Mach had asked:

“What share has every mass in the determination of direction ... in the law of inertia? No definite answer can be given by our experiences.”

Exact dragging of inertial axes by arbitrary cosmological energy currents in linear perturbation theory has been demonstrated for the first time in our papers [6, 7] for $K = 0$ and in the present paper for $K = \pm 1$.

J. Alternative, inequivalent postulates proposed by authors after Mach

Our presentation of Mach’s principle closely agrees with the one of Misner, Thorne, and Wheeler [37] and Weinberg [38]. However, many alternative and inequivalent postulates have been proposed under the heading “Mach’s principle” by many authors after E. Mach. An incomplete list of 10 inequivalent versions of Mach’s principle has been given by Bondi and Samuel [1], see also [8].

In the list of Bondi and Samuel the version “Mach 9” (“The theory contains no absolute elements”) is the postulate of the so-called “relativists”: Huyghens, Bishop Berkeley, Leibniz, Mach, and others. It was the starting point for Mach. See our “Test 3” in Sec. V H. — The

version “Mach 10” (“Overall rigid rotations of a system are unobservable”) is a necessary condition, formulated by Mach in his earliest writings [5] quoted in Sec. I A. See our “Test 4” in Sec. V H. Globally rigid rotations are discussed in Sec. V G.

In contrast the versions “Mach 1, 2, 6, 7” are not true in cosmological General Relativity, as already stated in [1]. — The version “Mach 3” (“local inertial frames influenced ...”) is irrelevant, because with only partial dragging one violates Mach’s postulates “no absolute element” in the input and “frame-invariance” in the solutions (see the quote of Mach in Sec. I A). — “Mach 4” (“The Universe is spatially closed”) is irrelevant, because we show that exact dragging also holds in open universes. — “Mach 5” (“The total angular momentum of the universe is zero”) is not true for that angular momentum which can be defined and is observable without an absolute element in the input (i.e. without prior knowledge of the local non-rotating frames), see Sec. I F and the end of Sec. II. — “Mach 8” (“ ρ/ρ_{crit} is of order unity”) is irrelevant, because we show in Secs. V D, V F that exact dragging also holds for $\rho/\rho_{\text{crit}} \gg 1$ and $\rho/\rho_{\text{crit}} \ll 1$.

With respect to this list of 10 versions of Mach’s principle we conclude: The criticisms “There are many formulations of Mach’s principle” and “It is not clear, what Mach meant” do not apply.

K. Outline

In Sec. II we derive the momentum constraint for (1) that momentum density which can be measured as an input without prior knowledge of the solution of the momentum constraint (the output), and for (2) the Laplacian Δ of de Rham-Hodge as opposed to the “rough Laplacian” ∇^2 .

In Sec. III we show that our \vec{B}_g field on a Minkowski background uniquely determines all nonzero components of the Riemann tensor.

To derive the *Green function* for the momentum constraint for the curved 3-spaces S^3 and H^3 , we need the solution in source-free regions, i.e. we need vorticity eigenfields of the Laplacian.

In a first step, in Sec. IV A, we derive *scalar* eigenfunctions of the Laplacian on S^3 and H^3 . Our new result: We give the radial functions of the scalar harmonics in a *simple, identical form for all three geometries* (E^3, S^3, H^3).

In Secs. IV B - IV D we give the expansion of vorticity fields (toroidal and poloidal) in terms of *vector spherical harmonics* $\vec{X}_{\ell m}^\pm$ and $Y_{\ell m} \vec{e}_\chi$. We show that the radial eigenfunctions of the Laplacian for *toroidal* vector fields are the spherical Bessel, Neumann, and Hankel functions generalized to S^3 and H^3 by the simple replacements $r \Rightarrow (\chi \text{ resp } R)$ given in Sec. I G.

In Sec. V we show that only toroidal \vec{A}_g -fields with $\ell = 1$ are relevant for the precession of gyroscopes at the origin and that the precession of a gyroscope cannot be caused by scalar or tensor perturbations. We then derive

the radial Green functions for the operator $(\Delta - \mu^2)$ acting on toroidal \vec{A}_g -fields with $\ell = 1$ on (E^3, S^3, H^3) . The final result is the gravitomagnetic \vec{B}_g -field at the position of the gyroscope, i.e. the precession of the gyroscope, expressed by the energy currents in the universe, and the equation which expresses exact dragging of inertial axes. We also show that Mach’s principle remains valid for perturbations of a universe with arbitrarily small total energy inside the Hubble radius, i.e. arbitrarily close to the Milne universe, and for a universe arbitrarily close to a de Sitter universe. — We derive the gravitomagnetic fields for the zero modes of the operator $(\Delta + 4K)$, i.e. for time-dependent globally rigid rotations of coordinates and FIDO axes in a FRW universe. — In Sec. V H we give *Six Fundamental Tests for Mach’s Principle*, all intimately related to Mach’s original formulation and all explicitly fulfilled by our solutions.

The Appendix gives the tools for the de Rham - Hodge Laplacian on vector fields in Riemannian 3-spaces, specifically the explicit equations connecting the calculus of differential forms with the elementary notation with curl etc. These tools are essential for cosmological gravitomagnetism.

II. THE MOMENTUM CONSTRAINT

The momentum constraint written in *local orthonormal tetrad* components is given in Eq. (5). It can be derived via two different general methods: (i) One can use the local orthonormal tetrad (LONB) method of Cartan from beginning to end. This method has been presented in Secs. V and VI of [7], where the connection coefficients for vorticity perturbations are given for FRW backgrounds with $K = (0, \pm 1)$, and $G_{\hat{0}\hat{i}}$ is derived for $K = 0$. We have used this method to derive Eq. (5) also for $K = \pm 1$, but we shall not present details again. (ii) Alternatively one can, in a first step, take the detour via the standard coordinate-component method with Christoffel symbols to obtain the momentum constraint in a *coordinate basis* with T_i^0 as the source. R_i^0 has been derived in Bardeen [21], Eq. (A2b), and in Kodama and Sasaki [28], Eq. (D.14b). In a second step one applies the basis transformation from the coordinate basis to the local orthonormal basis. — A modification of method (ii) is quicker: The basis transformation at the end of method (ii) can be replaced by taking Bardeen’s momentum constraint and moving the difference $(T_i^0 - T_i^{\hat{0}})$, which is meaningful at the level of linear perturbation theory, to the left-hand side of the momentum constraint.

The momentum constraint for R_i^0 of Bardeen and of Kodama and Sasaki reads in our notation

$$(\nabla^2 + 2K/a_c^2)A_g^i = 16\pi G_N(\rho_0 + p_0)v_c^i, \quad (12)$$

where a_c denotes the curvature scale, and ∇^2 refers to the physical scale. The 3-vector index i in A_g^i and v_c^i is lowered and raised by the 3-metric. — The zero modes of the

operator $(\Delta + 4K/a_c^2)$ correspond to time-dependent rigid rotations of the coordinates as demonstrated in Sec. V G. For non-zero modes of $(\Delta + 4K/a_c)$, our gravitomagnetic vector potential A_g^i , which is equal to our shift vector field β^i , is directly proportional to Bardeen's gauge-invariant amplitude Ψ (a combination of shift amplitude and shear amplitude),

$$A_g^i(x, t) = \beta^i(x, t) = -\Psi(t)Q^{(1)i}(x), \quad (13)$$

for a given wave number and polarization. For zero modes there is no shear, and a nonvanishing gauge-invariant amplitude does not exist. — Similarly for Bardeen's gauge-invariant velocity amplitude v_c

$$v_c^i(x, t) = v_c(t)Q^{(1)i}(x). \quad (14)$$

To convert Bardeen's momentum constraint Eq. (12) to our momentum constraint, the *first step* is replacing the “rough Laplacian” $\nabla^2 = \nabla^i \nabla_i$ by “the” Laplacian Δ (de Rham-Hodge Laplacian) via the *Weitzenböck* formula. With the Ricci tensor of the 3-space for FRW the Weitzenböck formula gives Eq. (A.19),

$$(\Delta - \nabla^2)\vec{A} = -(2K/a_c^2)\vec{A}. \quad (15)$$

Hence Bardeen's momentum constraint written in terms of the de Rham-Hodge Laplacian is

$$(\Delta + 4K/a_c^2)A_g^i = 16\pi G_N(\rho_0 + p_0)v_c^i. \quad (16)$$

The *second step* is replacing the gauge-invariant matter velocity amplitude v_c , used by Bardeen, by his other gauge-invariant amplitude v_s , used by us. In our gauge, which is shear-free, dx^i/dt is equal to Bardeen's gauge-invariant v_s^i .

The replacement of v_c by v_s is necessary, because the field with the amplitude v_c , used by Bardeen, is directly related to the angular velocity of matter *relative to spin axes of local gyroscopes* at the sources throughout the universe. This is *not* a measurable input without *prior knowledge of the solution of Einstein's equations*, i.e. of g_{0i} , \vec{A}_g , and \vec{B}_g . — In contrast, the field with the amplitude v_s , used by us, is directly related to the angular velocity of matter measured relative to geodesics (on Σ_t) which start at our position and go in the directions of the chosen spatial axes along our world line (resp geodesics to asymptotic quasars in an open asymptotically unperturbed FRW universe). The input needed in the context of Mach's principle is the field with the amplitude v_s . — The difference between Bardeen's two gauge-invariant velocity amplitudes is

$$(v_c - v_s) = -\Psi. \quad (17)$$

The Ψ term is a geometric output, hence it must be moved to the left-hand side, the geometric side (output side), of the momentum constraint. For the prefactors

on Bardeen's right-hand side we use the “second” Friedmann equation,

$$(dH/dt) - \frac{K}{a_c^2} = -4\pi G_N(\rho + p). \quad (18)$$

Combining the first and second steps gives the new operator on the left-hand side (geometric output side) of the momentum constraint: $[\Delta + (2 + 2 - 4)K/a_c^2 - \mu^2] = [\Delta - \mu^2]$, i.e. the three *curvature terms cancel*. The new term on the right-hand side (measured input side) is our energy current \vec{J}_ϵ . This gives Eq. (5).

From Eqs. (17, 13) we obtain $v_s^i = dx^i/dt = v_c^i - A_g^i$. Therefore the difference between the momentum densities $(\rho + p)v_c^i$, used by Bardeen, and $(\rho + p)v_s^i$, used by us, is analogous to the important difference between the *canonical* momentum $p_i = \partial L/\partial \dot{x}^i$ and the *kinetic* momentum $k^i = m dx^i/dt$ in Lagrangian mechanics for point particles of charge q in an electromagnetic field,

$$k^i = p^i - qA^i, \quad (19)$$

$$(\rho + p)v_s^i = (\rho + p)[v_c^i - A_g^i], \quad (20)$$

where the 3-indices are raised and lowered by the 3-metric. The *kinetic* (resp. *canonical*) momentum *is* (resp. *is not*) measurable without prior knowledge of A^i and A_g^i . The kinetic momentum density is equal to the LONB components $T_i^0 = (\rho + p)dx_i/dt$, while the canonical momentum is equal to the coordinate-basis components $T_i^0 = (\rho + p)g_{ij}^{(3)}[dx^j/dt + A_g^j]$.

To prove the postulate formulated by Mach (exact dragging of inertial axis directions) the kinetic momentum, resp the kinetic angular momentum of matter is the appropriate input. The canonical angular momentum (e.g. around the z -axis: T_ϕ^0), cannot be used, because its measurement resp. definition needs an absolute element in the input. — More details on kinetic versus canonical momentum and angular momentum in Sec. VIII of [7].

A cosmological term in Einstein's equations is diagonal in a LONB, hence a cosmological constant cannot appear in the momentum constraint in LONB components. If a cosmological constant Λ is absorbed in ρ and p , one has $(p_\Lambda/\rho_\Lambda) = -1$, and evidently Λ cannot make a contribution to $\vec{J}_\epsilon = (\rho + p)\vec{v}$.

III. RIEMANN TENSOR UNIQUELY GIVEN BY OUR \vec{B}_g FIELD

In this section we show that our choice of FIDOs and hence our (\vec{E}_g, \vec{B}_g) fields are singled out uniquely. We specialize to gravitomagnetism on a Minkowski background, since the material in this section will not be needed later, although it is very important: *Our specific choice of FIDOs is singled out by the property that only with our choice of FIDOs the (\vec{E}_g, \vec{B}_g) -fields uniquely determine the Riemann tensor and the gravito-electric and*

gravito-magnetic *tidal tensors* $(\mathcal{E}_{ij}, \mathcal{B}_{ij})$ for vector perturbations,

$$-R_{i\hat{0}\hat{j}\hat{0}} = \mathcal{E}_{ij} = E_{(ij)}^g, \quad (21)$$

$$-2(*R)_{i\hat{0}\hat{j}\hat{0}} \equiv \varepsilon_{i\hat{p}\hat{q}} R_{\hat{p}\hat{q}\hat{j}\hat{0}} = \mathcal{B}_{ij} = B_{(ij)}^g, \quad (22)$$

$$R_{\hat{k}\hat{0}} \equiv \delta_{\hat{r}\hat{s}} R_{\hat{k}\hat{r}\hat{0}\hat{s}} = \frac{1}{2}(\text{curl} \vec{B}_g)_{\hat{k}}. \quad (23)$$

The purely spatial Riemann tensor is zero for vector perturbations, i.e. the intrinsic geometry of each slice Σ_t remains unperturbed, as explained at the beginning of Sec. IB. — A vertical bar denotes the covariant derivative in 3-space, and a round bracket around two indices denotes symmetrization. — The equation for $R_{\hat{k}\hat{0}}$ is Eq. (35) of our companion paper [7].

The *first* equalities in Eqs. (21, 22) directly follow from the operational definition of the Riemann tensor: (1) $R_{i\hat{0}\hat{j}\hat{0}}$ gives the *relative acceleration* of neighboring free-falling, quasistatic particles (geodesic deviation), i.e. the *gravito-electric tidal field* \mathcal{E}_{ij} . The tidal 3-tensors $(\mathcal{E}_{ij}, \mathcal{B}_{ij})$ must be traceless by definition. The 3-trace of $R_{i\hat{0}\hat{j}\hat{0}}$ is a 3-scalar, therefore it is zero in the vector sector. — (2) The Hodge-star dual $(-R)_{i\hat{0}\hat{j}\hat{0}} \equiv -\frac{1}{2}\varepsilon_{i\hat{p}\hat{q}} R_{\hat{p}\hat{q}\hat{j}\hat{0}} \equiv \frac{1}{2}\varepsilon_{i\hat{p}\hat{q}} R_{\hat{p}\hat{q}\hat{j}\hat{0}}$ gives the *relative precession* of neighboring quasistatic gyroscopes' spins. The 3-trace of $(\varepsilon_{i\hat{p}\hat{q}} R_{\hat{p}\hat{q}\hat{j}\hat{0}})$ is zero because of the cyclic identity of the Riemann tensor, $R_{abcd} + R_{acdb} + R_{adbc} \equiv 0$. Replacing the relative precession by the *gravito-magnetic tidal field* \mathcal{B}_{ij} gives a factor $(-\frac{1}{2})$ according to Eq. (4) for any choice of FIDOs.

The *second* equalities in Eqs. (21-23) are only true for our (\vec{E}_g, \vec{B}_g) -fields, defined via *our specific choice of FIDOs*. Our FIDOs are *singled out* by the property that all 13 nonzero Riemann components are *uniquely determined* by spatial derivatives of our 2 vector fields (\vec{E}_g, \vec{B}_g) . If one also uses time-derivatives of the \vec{B}_g field and uses $\text{curl} \vec{E}_g = -\partial_t \vec{B}_g$ from Eq. (29) in [7], one concludes that *our* \vec{B}_g field by itself *determines* all 13 nonzero components of the *Riemann* tensor: Our \vec{B}_g field gives an efficient and physically transparent way to represent the entire information in the Riemann tensor.

The *Weyl tensor* $C_{\alpha\beta\gamma\delta}$ (defined by having no non-trivial 4-contractions) and its Hodge dual $*C$, have been used in the literature [39, 40, 41] to define the electric and magnetic tidal 3-tensors by contracting twice with a time-like unit vector. However, our tidal tensors have a *different normalization*, because tidal tensors must be directly connected to relative acceleration and relative precession (and hence to relative gravito-magnetic fields). The normalization of our tidal tensors in terms of the Weyl tensor is given by $C_{i\hat{0}\hat{j}\hat{0}} = \frac{1}{2}\mathcal{E}_{ij}$ and $*C_{i\hat{0}\hat{j}\hat{0}} = -\frac{1}{2}\mathcal{B}_{ij}$. The factor $\frac{1}{2}$ in the first equation comes from $C_{i\hat{0}\hat{j}\hat{0}} = \frac{1}{2}R_{i\hat{0}\hat{j}\hat{0}}$ valid for *vector* perturbations. The factor $(-\frac{1}{2})$ in the second equation comes from the conversion from the relative precession (given by the Riemann tensor) to the relative gravito-magnetic field.

The *curvature invariant* is $\mathbf{R} \cdot \mathbf{R} \equiv R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = 8[\mathcal{E}_{ij}\mathcal{E}_{ij} - \frac{1}{4}\mathcal{B}_{ij}\mathcal{B}_{ij}]$, and the curvature pseudoinvariant is $*\mathbf{R} \cdot \mathbf{R} \equiv (\frac{1}{2}\varepsilon_{\alpha\beta\rho\sigma} R^{\rho\sigma}_{\gamma\delta}) R^{\alpha\beta\gamma\delta} = -4\mathcal{E}_{ij}\mathcal{B}_{ij}$. These invariants are *not useful* for first-order perturbation theory, because local Lorentz transformations of \mathcal{E}_{ij} and \mathcal{B}_{ij} are *second-order* in perturbations.

The connection in the opposite direction, finding our \vec{B}_g field from the Ricci tensor: Any vector field in the 3-vector sector, i.e. with zero divergence, is the sum of a field determined by its curl plus a harmonic field, which in Euclidean 3-space is a homogeneous field, because we require that the \vec{B}_g -fields are not singular at spatial infinity. Therefore the Ricci tensor determines our \vec{B}_g -field up to a homogeneous \vec{B}_g -field, which is equivalent to a time-dependent spatially rigid rotation of all FIDOs according to Eq. (4). — Our solutions of the equations of cosmological gravitomagnetism on FRW backgrounds with $K = (0, \pm 1)$, e.g. Eq. (7), are *form-invariant* under time-dependent spatially rigid rotations, see Sec. IH. The exact dragging of inertial axes, Eq. (9), is *independent of the choice of FIDOs*.

Before Einstein's General Relativity, gravitomagnetic and gravitoelectric fields were discussed by Oliver Heaviside [42] based on a gravitational-electromagnetic analogy. With his equations Heaviside could have derived the partial dragging of inertial axes by a rotating star (Lense-Thirring effect), although a factor 4 would have been missing.

IV. EIGENFUNCTIONS OF THE LAPLACIAN FOR SCALAR FIELDS AND VORTICITY FIELDS IN (E^3, S^3, H^3)

Cosmological vorticity fields are of a different type from vorticity fields typical in fluid dynamics and condensed matter physics with their vortex lines (line singularities) and fields often idealized with cylindrical symmetry. In cosmology we consider vorticity fields (\equiv vector fields with vanishing divergence) which are *regular*, and we expand them in a *spherical basis*. These same fields also occur in the multipole expansion of electromagnetic radiation fields [43]. In this section we generalize these fields to the *curved 3-spaces* S^3 and H^3 . — In the first paragraph of Sec. IV B we explain, why the methods used in textbooks and in the literature for CMB-analyses are not adapted for our purpose.

Our aim is to solve the momentum constraint with its operator $(\Delta - \mu^2)$, i.e. the inhomogeneous Helmholtz equation. As a preparation for the Green function of $(\Delta - \mu^2)$ we derive the eigenfunctions of the Laplacian for (E^3, S^3, H^3) inside resp. outside a shell with sources. First we do this for scalar fields.

A. Eigenfunctions of the Laplacian for scalar fields in (E^3, S^3, H^3)

Our new results in this subsection: We give the radial functions of the scalar harmonics in a *simple, identical form* for all three geometries (E^3, S^3, H^3) , e.g. Eq. (30), a form which is familiar from the spherical Bessel functions for elementary wave physics in Euclidean 3-space. This is in contrast to the complicated derivations and results in the literature. — In Subsec. IV D we shall show that the radial functions are also identical for scalar and *toroidal vorticity* harmonics, if and only if one uses those vector spherical harmonics, $\vec{X}_{\ell m}^-$, which have point-wise norm and LONB components independent of R for fixed (θ, ϕ) .

Since the momentum constraint Eq. (5) is an elliptic equation (no partial time-derivatives), we consider, in this and all following sections, a FRW universe with $K = \pm 1$ at a *given time* with *units of length* chosen such that the curvature radius is $a_c = a = 1$ at the given time. We also include the case of FRW with $K = 0$ (Euclidean 3-space E^3) at a given time with the scale factor $a = 1$,

$$a = 1 \quad (24)$$

$$ds^2 = d\chi^2 + R_K^2(\chi) [d\theta^2 + \sin^2 \theta d\phi^2] \quad (25)$$

$$R_K(\chi) = \{\chi, \sin \chi, \sinh \chi\}, \quad K = \{0, +1, -1\}. \quad (26)$$

The following relations will be useful,

$$R'' = -KR, \quad R'^2 = 1 - KR^2, \quad (27)$$

where $' = d/d\chi$, and R is short for $R_K(\chi)$.

For scalar fields the angular eigenfunctions of the Laplacian Δ are spherical harmonics $Y_{\ell m}(\theta, \phi)$ with eigenvalues $-\ell(\ell+1)/R^2$. — The radial eigenfunctions of the Laplacian are the generalization to $K = \pm 1$ of the spherical Bessel functions $j_\ell(qr)$, which are the radial eigenfunctions in Euclidean 3-space. These *generalized spherical Bessel functions*, the hyperspherical Bessel functions, will be denoted in this paper by $\tilde{j}_{q\ell}^{(K)}(\chi)$ to emphasize the close correspondence to $\tilde{j}_{q\ell}^{(K=0)}(\chi) \equiv j_\ell(q\chi)$. We shall always denote the *eigenvalues* of the de Rham-Hodge Laplacian Δ by $(-k^2)$. For Δ acting on scalar eigenfunctions and on vector or tensor eigenfunctions in the scalar sector we define q by

$$\text{scalar sector:} \quad q^2 = k^2 + K. \quad (28)$$

It will turn out that q is the *wave number* in the *radial oscillations* $\sin(q\chi)$ in Eqs. (30, 31, 32).

The radial part of the Laplace operator acting on a scalar function $f(\chi)$ is $\Delta f = R^{-2} \frac{d}{d\chi} (R^2 \frac{d}{d\chi} f)$. Using $R^{-1} \frac{d^2}{d\chi^2} R = -K$ we obtain $\Delta f = R^{-1} \frac{d^2}{d\chi^2} (Rf) + Kf$. Hence the radial eigenfunctions for scalar harmonics satisfy

$$\frac{1}{R} \frac{d^2}{d\chi^2} (R \tilde{j}_{q\ell}^{(K)}) = [-q^2 + \frac{\ell(\ell+1)}{R^2}] \tilde{j}_{q\ell}^{(K)}. \quad (29)$$

The solutions regular at the origin are

$$\tilde{j}_{q\ell}^{(K)}(\chi) = R^\ell \left(-\frac{1}{R} \frac{d}{d\chi}\right)^\ell \left(\frac{\sin q\chi}{qR}\right), \quad (30)$$

which we call *generalized spherical Bessel functions*. Writing the radial equation and the solutions in this form gives a *unified notation* for all three geometries (E^3, S^3, H^3) . Compared to spherical Bessel functions (familiar from wave physics in Euclidean 3-space) the only changes are the replacements of the *independent variable*, $r \rightarrow \chi$, in the radial oscillations and in the derivative, and the replacement of the *denominators and prefactors*, $r \rightarrow R(\chi)$. — From Eq. (30) we obtain the generalized spherical Bessel functions $\tilde{j}_{q\ell}^{(K)}(\chi)$ for $\ell = (0, 1)$,

$$\tilde{j}_{q,\ell=0}^{(K)}(\chi) = \frac{1}{qR} \sin(q\chi), \quad (31)$$

$$\tilde{j}_{q,\ell=1}^{(K)}(\chi) = -\frac{1}{q^2} \frac{d}{d\chi} \left[\frac{1}{R} \sin(q\chi) \right]. \quad (32)$$

The generalized spherical Neumann functions are obtained from Eqs. (30, 31, 32) by the replacements $\sin(q\chi) \rightarrow -\cos(q\chi)$ and $\cos(q\chi) \rightarrow +\sin(q\chi)$. The generalized spherical Hankel functions are given by $\tilde{h}^{(1,2)} = \tilde{j} \pm i\tilde{n}$. — Relevant for Mach's principle are the spherical Bessel and Hankel functions for $\ell = 1$ generalized to $K = \pm 1$. They are given by Eq. (32) and

$$\tilde{h}_{q,\ell=1}^{(1,2)(K)}(\chi) = \pm i \frac{1}{q^2} \frac{d}{d\chi} \left[\frac{1}{R} \exp(\pm i q\chi) \right]. \quad (33)$$

The eigenfunctions of the Laplacian will be used in the context of Green functions with sources at $\chi = \chi_s$, hence they will be used only for $0 \leq \chi < \chi_s$ resp. for $\chi > \chi_s$. Therefore there are *no geometric restrictions on the eigenvalues* in our context.

The *simplicity* of the elementary functions (32, 33), their *identical form* for all three geometries, the *familiar appearance* from elementary wave physics in Euclidean 3-space, and the *straightforward derivation* are in contrast to the complicated derivations for the hyperspherical Bessel functions $\Phi_\ell^\nu(\chi)$ via the associated Legendre functions $P_{-1/2+\beta}^{-1/2-\ell}(\cos y)$ discussed extensively in [44] and used by [45].

B. Vector spherical harmonics

In this subsection we introduce the 3-dimensional vector spherical harmonics $(\vec{X}_{\ell m}^\pm, \vec{e}_\chi Y_{\ell m})$ and the Regge-Wheeler harmonics $(\vec{x}_{\ell m}^\pm, \vec{d}_\chi Y_{\ell m})$, and we obtain their curls and divergences in Eqs. (44) - (49), which are valid and *form-identical* not only for the 3-spaces (S^3, H^3, E^3) , but for *any spherically symmetric Riemannian 3-space*. Our curl- and div-equations for this *general case* are *very much simpler* than the equations in the literature for the special case of Euclidean 3-space. — The general and

simple equations (44) - (49) are *all that's needed* to construct vorticity fields (divergenceless) and to explicitly write down the de Rham-Hodge Laplacian on vorticity fields, $\Delta \vec{V} = -\text{curl curl } \vec{V}$, for *any* Riemannian 3-space with spherical symmetry. — Textbooks usually present the vector spherical harmonics of [46, 47], which are not adapted to cosmological perturbation theory, because they mix the vector and the scalar sector, see the end of this subsection. — The spin- s spherical harmonics used in CMB analyses [31] are not adapted to our problem, because they mix parities, i.e. the toroidal and poloidal sectors of vector spherical harmonics, see the end of this subsection and Sec. IV C.

Vector spherical harmonics $\vec{X}_{\ell m}^{\pm}(\theta, \phi)$ form a basis for vector fields tangent to the 2-sphere. In the notation of differential forms, the vector spherical harmonics of *Regge and Wheeler* [48], which we denote by a *lower case* $\tilde{x}_{\ell m}^{\pm}$, are given by

$$\tilde{x}_{\ell m}^{+} \equiv dY_{\ell m}, \quad \tilde{x}_{\ell m}^{-} \equiv -^{(2)} * dY_{\ell m}, \quad (34)$$

i.e. the *gradient* of $Y_{\ell m}$, and its *Hodge dual* on the 2-sphere, $^{(2)} *$. In component notation Eq. (34) gives

$$(x_{\ell m}^{+})_{\alpha} = \partial_{\alpha} Y_{\ell m}, \quad (35)$$

$$(x_{\ell m}^{-})_{\alpha} = -\varepsilon_{\alpha\beta} g^{\beta\gamma} \partial_{\gamma} Y_{\ell m}, \quad (36)$$

where $\alpha = (\theta, \phi)$, $\varepsilon_{\alpha\beta} = \sqrt{^{(2)}g} \varepsilon_{\hat{\alpha}\hat{\beta}}$, $\varepsilon_{\hat{\theta}\hat{\phi}} \equiv +1$. In the combination $\varepsilon_{\mu\nu} g^{\nu\lambda} = \varepsilon_{\mu}^{\lambda}$ the radius R of the 2-sphere drops out, $\varepsilon_{\theta}^{\phi} = (\sin \theta)^{-1}$, $\varepsilon_{\phi}^{\theta} = -(\sin \theta)$.

The vector harmonics of Regge and Wheeler have *covariant components* (1-form components) which are independent of the radial coordinate χ ,

$$\partial_{\chi}(x_{\ell m}^{\pm})_{\alpha} = 0, \quad \alpha = (\theta, \phi). \quad (37)$$

In contrast, the '*physical*' vector spherical harmonics are denoted by the *upper case* $\vec{X}_{\ell m}^{\pm}$. They are used in the literature on classical electrodynamics [43] and called pure-spin vector harmonics by Thorne [49]. They are defined with an R -factor which makes the *radial covariant derivative* vanish,

$$\vec{X}_{\ell m}^{\pm} \equiv R \tilde{x}_{\ell m}^{\pm}, \quad (38)$$

$$\nabla_{\chi} \vec{X}_{\ell m}^{\pm} = 0. \quad (39)$$

This is equivalent to requiring that the *point-wise norm* $g(\vec{X}_{\ell m}^{\pm}, \vec{X}_{\ell m}^{\pm})$ and the *LONB components* of $\vec{X}_{\ell m}^{\pm}$ are independent of R for fixed (θ, ϕ) .

We rewrite the 2-dimensional Hodge dual of Eq. (34) as $^{(2)} * \vec{\nabla} Y_{\ell m} = -\vec{e}_{\chi} \times \vec{\nabla} Y_{\ell m}$. The generator of rotations, the angular momentum operator of wave mechanics in units of \hbar , is $\vec{L} = -i(\vec{R} \times \nabla)$, where $\vec{R} \equiv R(\chi)\vec{e}_{\chi}$. Hence

$$\vec{X}_{\ell m}^{+} = R \vec{\nabla} Y_{\ell m} \quad (40)$$

$$\vec{X}_{\ell m}^{-} = i \vec{L} Y_{\ell m} = (\vec{R} \times \vec{\nabla}) Y_{\ell m} = \vec{e}_{\chi} \times \vec{X}_{\ell m}^{+} \quad (41)$$

$$\vec{e}_{\chi} Y_{\ell m}, \quad (42)$$

where we have also listed the third basis field needed for 3 dimensions, $\vec{e}_{\chi} Y_{\ell m}$. The triple $(\vec{X}^{+}, \vec{X}^{-}, \vec{e}_{\chi})$ is point-wise orthogonal and has positive orientation. \vec{X}^{+} and \vec{X}^{-} have the same norm point-wise.

The *parity* is $P = (-1)^{\ell}$ for $\vec{X}_{\ell m}^{+}$ and $\vec{e}_{\chi} Y_{\ell m}$, while $P = (-1)^{\ell+1}$ for $\vec{X}_{\ell m}^{-}$.

All three basis fields $\{\vec{X}_{\ell m}^{\pm}, \vec{e}_{\chi} Y_{\ell m}\}$ are eigenfunctions of the total angular momentum operators $\{J^2, J_z\}$ with eigenvalues $\{\ell(\ell+1), m\}$. This follows because exterior differentiation \vec{d} , taking the Hodge dual $(*)$, and multiplication with \vec{e}_{χ} commute with rotations of the total system.

The *normalization* and *orthogonality* relation on the 2-sphere at any R is

$$\begin{aligned} (\vec{X}_{\ell m}^{(p)}, \vec{X}_{\ell' m'}^{(p')}) &= \int d\Omega < \vec{X}_{\ell m}^{(p)}, \vec{X}_{\ell' m'}^{(p')} > \\ &= \ell(\ell+1) \delta_{\ell\ell'} \delta_{mm'} \delta_{pp'}. \end{aligned} \quad (43)$$

This relation is crucial for the projection of a general velocity field on the term with $(\ell=1, P=+)$. The global (Hilbert space) scalar product on the 2-sphere is denoted by (\vec{V}, \vec{W}) , and $< \vec{V}, \vec{W} >$ denotes the point-wise scalar product $g(\vec{V}^*, \vec{W})$, where the superscript $*$ stands for the complex conjugate.

Computations of curl and div and *results* are much *simpler* for Regge-Wheeler harmonics $\tilde{x}_{\ell m}^{\pm}$ than for the 'physical' harmonics $\vec{X}_{\ell m}^{\pm}$. — Also the basis fields $[\vec{e}_{\chi} R^{-2} Y_{\ell m}]$ are computationally much simpler than the basis fields $[\vec{e}_{\chi} Y_{\ell m}]$, because the divergence of the former vanishes apart from a δ -function at the origin, which turns out to be irrelevant in our context,

$$\text{div } \tilde{x}_{\ell m}^{+} = -\frac{\ell(\ell+1)}{R^2} Y_{\ell m} \quad (44)$$

$$\text{curl } \tilde{x}_{\ell m}^{+} = 0 \quad (45)$$

$$\text{div } \tilde{x}_{\ell m}^{-} = 0 \quad (46)$$

$$\text{curl } \tilde{x}_{\ell m}^{-} = -\frac{\ell(\ell+1)}{R^2} Y_{\ell m} \vec{e}_{\chi} \quad (47)$$

$$\text{div } [Y_{\ell m} \frac{\vec{e}_{\chi}}{R^2}] = 0 \quad (48)$$

$$\text{curl } [Y_{\ell m} \frac{\vec{e}_{\chi}}{R^2}] = -\frac{1}{R^2} \tilde{x}_{\ell m}^{-}. \quad (49)$$

If we had computed the divergence and curl of $\vec{X}_{\ell m}^{\pm}$ and $Y_{\ell m} \vec{e}_{\chi}$, there would have been twice as many terms, and the terms would have been more complicated.

Computations of $\text{curl } \vec{V} = \vec{\nabla} \times \vec{V}$ and $\text{div } \vec{V} = \vec{\nabla} \cdot \vec{V}$, where $\vec{\nabla}$ is the covariant derivative, are made very simple with the calculus of *differential forms*, where no Christoffel symbols are needed, see Eqs. (A.2, A.5).

The *spin-s spherical harmonics* of Newman and Penrose [49, 50, 51], which are widely used for cosmic microwave anisotropies [31], are *not adapted* to our problem, because they *mix parities*, i.e. they mix toroidal and

poloidal vorticity fields (see Sec. IV C), while the dragging of inertial axis-directions is caused by energy-flows with $J^P = 1^+$, i.e. in the toroidal sector (see Sec. V A).

In most textbooks vector spherical harmonics are constructed by coupling a definite *orbital angular momentum* with *spin-1 basis vectors* to obtain a definite total angular momentum via Clebsch-Gordan coefficients [46, 47, 49]. These basis states are *not adapted* to cosmological perturbation theory, because they *mix* the *poloidal vorticity sector* and the *scalar sector*, see the following subsection.

C. Toroidal and poloidal vorticity fields

For 3-dimensional vorticity fields the 3-divergence must be zero. The vector spherical harmonics $\vec{x}_{\ell m}^-$ for $m = 0$ are vector fields purely along the direction of $\pm \vec{e}_\phi$ and independent of ϕ . Therefore the $\vec{x}_{\ell m}^-$ are named ‘toroidal’, and evidently their divergence is automatically zero. The toroidal harmonics with $m \neq 0$ are generated from those with $m = 0$ by rotations, and their divergence is again zero, as also seen from Eq. (46).— Multiplication with a radial function $g^{\text{tor}}(\chi)$ gives the 3-dimensional toroidal vorticity fields,

$$\vec{V}_{\ell m}^{\text{tor}}(\chi, \theta, \phi) = g_{\ell}^{\text{tor}}(\chi) \vec{x}_{\ell m}^-(\theta, \phi). \quad (50)$$

The *3-divergence* is *not zero* for the vector spherical harmonics $\vec{x}_{\ell m}^+$ by themselves, $\text{div} \vec{x}_{\ell m}^+ \neq 0$. To obtain vector fields \vec{V} with zero 3-divergence we must add a part perpendicular to S^2 , $\vec{e}_\chi F(\chi, \theta, \phi)$, where F can be expanded in scalar spherical harmonics $Y_{\ell m}$. Hence the 3-dimensional *poloidal vorticity fields* can be written

$$\vec{V}_{\ell m}^{\text{pol}}(\chi, \theta, \phi) = g_{\ell}^{(\text{pol}, \text{tg})} \vec{x}_{\ell m}^+ + g_{\ell}^{(\text{pol}, \text{rad})} [Y_{\ell m} R^{-2} \vec{e}_\chi]. \quad (51)$$

Both terms have $P = (-1)^\ell$. For $m = 0$ this equation gives vorticity fields \vec{V} with only radial and θ -components, since $\vec{x}_{\ell, m=0}^+$ points along the meridians. Hence the name ‘poloidal’ for these vorticity fields.

The condition for vanishing divergence of the poloidal vorticity fields is evaluated using the vector identity $\text{div}(g\vec{V}) = g \text{div} \vec{V} + \vec{V} \cdot \text{grad} g$ and Eqs. (44, 48),

$$\frac{d}{d\chi} g_{\ell}^{(\text{pol}, \text{rad})} = \ell(\ell+1) g_{\ell}^{(\text{pol}, \text{tg})}. \quad (52)$$

Again, the equations in this subsection are valid for *any* Riemannian 3-space with spherical symmetry.

D. The radial functions for vorticity harmonics

Relevant for rotational frame-dragging are energy currents and vector potentials in the *toroidal* sector as demonstrated in Sec. V A. Hence we now focus on the toroidal eigenfunctions of the Laplacian. — *Poloidal* energy currents cannot contribute to the precession of a

gyroscope at the origin. Therefore poloidal eigenfunctions of the Laplacian are not considered in this paper, but they will be needed and presented in a forthcoming paper, “The other half of Mach’s principle: Frame dragging for linear acceleration” [52].

In this subsection we derive the following important and simple *result*:

1. In the eigenfunctions of the Laplacian for *toroidal vorticity* fields on (S^3, H^3, E^3) : The *physical* vector spherical harmonics $\vec{X}_{\ell m}^-$ are multiplied by *radial functions* which are *identical* to those for scalar harmonics, i.e. the *generalized spherical Bessel functions* $\tilde{j}_{q\ell}^{(K)}(\chi)$ of Eq. (30) resp. the corresponding Neumann and Hankel functions.
2. For *toroidal vorticity* harmonics we have the relation $q = k$:

$$(\Delta + q^2)[\tilde{j}_{q\ell}^{(K)}(\chi) \vec{X}_{\ell m}^-(\theta, \phi)] = 0. \quad (53)$$

In all sectors q denotes the *wave number in the radial oscillations* $\sin(q\chi)$, and $(-k^2)$ denotes the *eigenvalue of the de Rham-Hodge Laplacian*. Note that the relation for scalar harmonics is $q^2 = k^2 + K$, Eq. (28).

To prove this result, all that’s needed is the expression for the de Rham-Hodge Laplacian for vorticity fields, $\Delta = -\text{curl} \text{curl} \vec{A}$, and the simple Eqs. (44) - (49). — For the rest of this subsection we drop the subscripts (ℓ, m) . We start with a toroidal vorticity field $\vec{A}^{\text{tor}} = g_A^{\text{tor}} \vec{x}^-$ and evaluate its curl. Using the vector identity $\text{curl}(g\vec{v}) = g \text{curl} \vec{v} + (\text{grad} g) \times \vec{v}$ and Eq. (47) gives

$$\begin{aligned} \text{curl} \vec{A}^{\text{tor}} &= \text{curl}(g_A^{\text{tor}} \vec{x}^-) \\ &= -\left(\frac{d}{d\chi} g_A^{\text{tor}}\right) \vec{x}^+ - \ell(\ell+1) g_A^{\text{tor}} [Y R^{-2} \vec{e}_\chi]. \end{aligned} \quad (54)$$

In a second step we take the curl a second time, which gives minus the de Rham-Hodge Laplacian of \vec{A}^{tor} ,

$$\begin{aligned} \Delta \vec{A}^{\text{tor}} &= -\text{curl} \text{curl} \vec{A}^{\text{tor}} \\ &= \left[\frac{d^2}{d\chi^2} g_A^{\text{tor}} - \frac{\ell(\ell+1)}{R^2} g_A^{\text{tor}}\right] \vec{x}^-, \end{aligned} \quad (55)$$

where we have used Eqs. (45, 49). Up to here all formulae in Secs. IV B - IV D are valid for *any* Riemannian 3-space with spherical symmetry. We now specialize to (S^3, H^3, E^3) . We rewrite the toroidal vorticity fields in terms of the *physical* vector harmonics $\vec{X}_{\ell m}^- = R \vec{x}_{\ell m}^-$ and their associated radial functions $G^{\text{tor}} \equiv R^{-1} g^{\text{tor}}$, hence $g^{\text{tor}} \vec{x}^- = G^{\text{tor}} \vec{X}^-$. Comparing Eq. (55) rewritten in terms of G^{tor} with Eq. (29) gives the main result of this section, Eq. (53).

In contrast to [16] we have given a *simple, unified treatment* of all three cases, $K = (0, \pm 1)$, with *form-identical derivations and results*. — Ref. [16] discussed the solutions of the homogeneous momentum constraint (which are eigenfunctions of the Laplacian) without referring to the Laplacian.

V. MACH'S PRINCIPLE

A. Symmetries and Mach's principle

We treat *all* linear perturbation fields on *all* types of FRW backgrounds, $K = (\pm 1, 0)$, and *all* energy-momentum-stress tensors, i.e. *all* types of matter (also dark energy and a cosmological constant), not necessarily of the perfect-fluid form, and for *totally general field configurations* of energy currents $J_k^\varepsilon \equiv T_k^\varepsilon$. This is in striking contrast to the previous literature, which only treated the artificial situation of spherical shells of matter rotating rigidly around one given axis. — However the *mathematics* of totally general perturbations can be reduced to the mathematics of the special case of spherical shells of matter in rigid rotation around a given gyroscope with the help of *two theorems*, which are based on the symmetries relevant for Mach's principle, rotation and parity:

1. The *precession* of a *gyroscope* (relative to given local LONB axes) *cannot* be caused by *scalar* perturbations nor, in linear perturbation theory, by *tensor* perturbations. — In the vector sector the precession of a gyroscope can be caused only by energy-current fields with $J^P = 1^+$ relative to the given gyroscope's position, i.e. of the *toroidal* type and with $\ell = 1$.
2. On every mathematical spherical shell centered on the gyroscope considered: The field component (in the harmonic decomposition of the general energy-current field) which is toroidal and has $\ell = 1$ (relative to the gyroscope) is given by an *equivalent rigid rotation* with an *equivalent angular velocity* of matter $\vec{\Omega}_{\text{equiv}}^{\text{matter}}(\chi_s)$. The equivalent angular velocity of matter is given by the *global scalar product* of the \vec{J}_ε -field with the toroidal fields $\vec{X}_{\ell=1,m}^-$ on the shell with radius χ :

$$\int d\Omega < \vec{X}_{\ell=1,m}^{-*}, \vec{J}_\varepsilon(\chi, \theta, \phi) > \equiv (J_\varepsilon)_{\ell=1,m}^-(\chi) = -\sqrt{16\pi/3}(\rho_0 + p_0) R(\chi) [\Omega_m(\chi)]_{\text{equiv}}^{\text{matter}}, \quad (56)$$

where $< \dots, \dots >$ denotes the point-wise inner product, and $d\Omega$ is the element of solid angle, while Ω_m denotes spherical-basis components of the angular velocity. The transformation between the spherical and the Cartesian components of $\vec{\Omega}$ is given by $\Omega_{m=0} = \Omega_z$, and $\Omega_{m=\pm 1} = (\mp \Omega_x - i\Omega_y)/\sqrt{2}$.

The proof of theorems (1) uses the following facts: The precession of a gyroscope relative to given LONB axes, $d(S_i)/dt$, which equals the *torque* on the gyroscope, is an *axial vector*, $J^P = 1^+$. The precession of a gyroscope is caused locally by the gravitomagnetic field $\vec{B}_g(0)$, which is also an axial vector. — For *scalar perturbations* all fields are derived from scalar fields (via differentiation),

but this can only produce source-fields \vec{J}_ε in the natural parity sequence, $0^+, 1^-, 2^+$, etc, which cannot contribute to the precession. — For *tensor perturbations* (gravitational waves) all linear perturbations are given by a traceless, divergenceless 3-tensor, from which one cannot form an axial vector field to generate a \vec{B}_g field at the origin.

The proof of theorem (2) uses the following facts: For a general energy-momentum-stress tensor (not necessarily of the perfect-fluid type) the local center-of-mass velocity \vec{v} is given by $v_k \equiv (\rho_0 + p_0)^{-1} T_k^\varepsilon$ in linear perturbation theory, where ρ_0 and p_0 refer to the unperturbed FRW background. — For fixed χ , a toroidal velocity field $\vec{X}_{\ell=1,m}^-$ is a flow equivalent to a rigid rotation of a shell of matter.

The gravitomagnetic *vector potentials* \vec{A}_g relevant for the gyroscope's precession, must also be *toroidal* with $\ell = 1$. The *gravitomagnetic field* $\vec{B}_g = \text{curl} \vec{A}_g$, which causes the gyroscope's precession, must be *poloidal* with $\ell = 1$.

B. Green functions for $(\Delta - \mu^2)$ on vorticity fields in open FRW universes

We derive the Green function for the gravitational vector potential \vec{A}_g for $K = (-1, 0)$ generated by an energy-current source which is toroidal, has $\ell = 1$, $m = 0$, and is concentrated at the geodesic distance χ_s relative to the gyroscope considered. Such a source corresponds to a shell of cosmological matter in rigid rotation around the z -axis,

$$v^\phi = \Omega_s \delta(\chi - \chi_s). \quad (57)$$

The Green function for arbitrary ℓ is not needed to compute the dragging of inertial axes at the origin. The derivation for general ℓ is totally analogous to the case $\ell = 1$ treated here, and it gives the general solution of the momentum constraint for the toroidal sector.

As noted at the beginning of Sec. IV A, all equations in Secs. IV and V involve the perturbation quantities at one time, and at the chosen time we set the scale factor a equal to *one*.

Inside and outside the rotating shell, \vec{A} is a *toroidal vector eigenfield* of $(\Delta - \mu^2)$ according to the momentum constraint Eq. (5). The wave number q in the radial oscillations is $q = \pm i\mu$ according to Eq. (53). We choose the positive sign.

Inside the shell the radial function $G(\chi) = f(\chi)$ which multiplies $\vec{X}_{\ell=1,m}^-$ according to Eq. (53) is the generalized spherical Bessel function for $K = (\pm 1, 0)$, Eq. (32),

$$i\mu^2 \vec{J}_{q=i\mu, \ell=1}^{(K)}(\chi) = \frac{-d}{d\chi} \left[\frac{1}{R} \sinh(\mu\chi) \right]. \quad (58)$$

Note that any *regular* toroidal vector field must *vanish* at the origin.

Outside the shell the radial function for an *open* universe, $K = (-1, 0)$, which is *regular* at spatial infinity

for $q = i\mu$, is the generalized Hankel function of the first kind, Eq. (33), exponentially decreasing for $\chi \rightarrow \infty$,

$$\begin{aligned} -i\mu^2 \tilde{h}_{q=i\mu, \ell=1}^{(1)(K)}(\chi) &= \frac{-d}{d\chi} \left[\frac{1}{R} \exp(-\mu\chi) \right] \quad (59) \\ &= Y_\mu(\chi) = \text{Yukawa force.} \quad (60) \end{aligned}$$

Note that the Yukawa *potential* is $(1/R) \exp(-\mu\chi)$, while the Yukawa *force* is minus the gradient (the χ -derivative) of the potential. — For a closed universe this solution is not acceptable, because the $1/R$ factor makes it *singular* at $\chi = \pi$, the *antipodal point* to the origin. This will be treated in subsection V D.

The vector potential \vec{A}^{tor} is tangential to the shell, and it must be *continuous across the shell*, because the component of \vec{B} normal to the shell must be continuous, otherwise $\text{div} \vec{B} \neq 0$ on the shell. In an open universe, $K = (-1, 0)$, we obtain

$$f_A^{\text{tor}}(\chi, \chi_s) = N \tilde{j}(\chi_{<}) \tilde{h}^{(1)}(\chi_{>}), \quad (61)$$

where we have dropped the subscripts $\{q = i\mu, \ell = 1\}$ and the superscript $(K = -1, 0)$ for better readability. The arguments $\chi_{<}$ resp. $\chi_{>}$ denote the smaller resp. larger of the two arguments (χ, χ_s) , and χ_s refers to the radius of the shell.

The normalization factor N is determined by the strength of the source \vec{J}_ϵ on the shell: Einstein's $G_\phi^{\hat{0}}$ equation, Eq. (5), determines the discontinuity of the tangential component $B_{\hat{\theta}}$,

$$\begin{aligned} (\text{curl } \vec{B})_{\hat{\phi}}^{\text{sing}} &= \delta(\chi - \chi_s) \text{disc}[B_{\hat{\theta}}] = -16\pi G_N J_{\hat{\phi}}^\epsilon \\ &= -\delta(\chi - \chi_s) 16\pi G_N (\rho + p) \Omega_s R_s \sin \theta. \quad (62) \end{aligned}$$

The discontinuity of $B_{\hat{\theta}}$ is determined by the discontinuity of its radial function $[f_B^{(\text{pol}, \text{tg})}]$. For an open universe, $K = (-1, 0)$, Eqs. (52, 54, 61) give

$$\begin{aligned} \text{disc } [f_B^{(\text{pol}, \text{tg})}] &= -\text{disc} \left[\frac{d}{d\chi} f_A^{\text{tor}} \right] \\ &= -N \text{disc} \left[\frac{d}{d\chi} \{ \tilde{j}(\chi_{<}) \tilde{h}^{(1)}(\chi_{>}) \} \right]_s \\ &= -N W[\tilde{j}, \tilde{h}^{(1)}]_s, \quad (63) \end{aligned}$$

where we denote the Wronskian by $W[f, g] \equiv (fg' - f'g)$ with $\prime = \frac{d}{d\chi}$. Our Wronskian multiplied with R^2 is independent of χ , as can be seen from Eq. (29). Therefore we evaluate $R^2 W$ in the limit $\chi \rightarrow 0$. We use $\tilde{j}_{q=i\mu, \ell=1}^{(K)}(\chi) \rightarrow (i\chi\mu/3)(1+K/\mu^2)$ and $\tilde{h}_{q=i\mu, \ell=1}^{(K)}(\chi) \rightarrow (\mu\chi)^{-2}$ for $\chi \ll 1$, and we obtain

$$W[\tilde{j}_{q, \ell}^{(K)}, \tilde{h}_{q, \ell}^{(1)(K)}]_{q=i\mu}^{\ell=1} = \frac{1}{\mu R^2} (1 + \frac{K}{\mu^2}) \quad (64)$$

valid for $K = (\pm 1, 0)$ and for all R . We note that for $(\ell = 1, m = 0)$

$$(X_{\ell=1, m=0}^+)_{\hat{\theta}} = (X_{\ell=1, m=0}^-)_{\hat{\phi}} = -\sqrt{3/(4\pi)} \sin \theta, \quad (65)$$

and all other components are zero. We multiply $\text{disc}[f_B^{(\text{pol}, \text{tg})}]$ with $(X_{\ell=1, m=0}^+)_{\hat{\theta}}$ to obtain $\text{disc}[B_{\hat{\theta}}]$, which we insert in Eq. (62) to determine N ,

$$N = -(16\pi G_N)(\rho + p) R_s^3 \Omega_s \mu (1 + \frac{K}{\mu^2})^{-1} \sqrt{4\pi/3}. \quad (66)$$

This completes the computation of the Green function, Eq. (61), in an open universe, $K = (-1, 0)$, for the vector potential generated by the toroidal current \vec{J}_ϵ with $\ell = 1, m = 0$ at χ_s , specifically by the velocity field of Eq. (57).

The gravitomagnetic field \vec{B} at the origin, the position of the gyroscope, is obtained from Eq. (54). $\vec{B}_g(0)$ written for arbitrary orientation of $\vec{\Omega}_s$ is

$$\vec{B}(0) = -2\vec{\Omega}_{\text{gyro}}(0) = -\vec{\Omega}_s \frac{2}{3} [16\pi G_N (\rho + p)] R_s^3 Y_\mu(\chi_s), \quad (67)$$

valid in an open universe, $K = (-1, 0)$. This equation shows, how $\vec{\Omega}_s$, the angular velocity of the rotating shell, determines $\vec{\Omega}_{\text{gyro}}$, the angular velocity of precession of a gyroscope at the center of the rotating shell.

In Eq. (67) we denote the prefactor by $\tilde{\mu}^2$,

$$\tilde{\mu}^2 \equiv 16\pi G_N (\rho + p) \quad (68)$$

$$\mu^2 \equiv -4(dH/dt). \quad (69)$$

The connection between $\tilde{\mu}$ and μ is given by the (dH/dt) equation, which follows by subtracting the first from the second Friedmann equation,

$$\begin{aligned} (dH/dt) - \frac{K}{a_c^2} &= -4\pi G_N (\rho + p) \\ a_c \equiv 1 \Rightarrow \left(\frac{\mu}{2}\right)^2 + K &= \left(\frac{\tilde{\mu}}{2}\right)^2. \quad (70) \end{aligned}$$

Three length scales occur in our problem: the H -dot radius $(\mu/2)^{-1}$, the $(\rho + p)$ -radius $(\tilde{\mu}/2)^{-1}$, and the curvature radius $a_c \equiv 1$.

C. Exact dragging of inertial axes for perturbations of an open FRW universe

The main result of this paper: (1) The equations for “Dragging of Inertial Axes”, Eqs. (9, 10), are obtained directly from Eq. (67) by integrating over all shell radii. (2) The equation, which shows that there is *exact dragging* (as opposed to partial dragging) of local inertial axis directions, is the statement that the weight function $W(\chi)$ has its *normalization* equal to *unity*, Eq. (11).

We now give the proof that the normalization of the weight function is indeed equal to 1 for the case of an open universe, $K = (-1, 0)$. First one does a partial integration of $(d/d\chi)$ in the Yukawa force. This gives $W_{\text{tot}} = \tilde{\mu}^2 \int d\chi R R' \exp(-\mu\chi)$. A partial integration of the factor $\exp(-\mu\chi)$ gives $W_{\text{tot}} = \tilde{\mu}^2 \int d\chi (1 - 2KR^2) \mu^{-1} \exp(-\mu\chi)$,

where we have used Eq. (27). If instead of the last step, one performs a partial integration of the factor (RR') , one obtains $W_{\text{tot}} = \tilde{\mu}^2 \int d\chi (R^2/2) \mu \exp(-\mu\chi)$. Adding μ^2 times the first expression plus $(4K)$ times the second expression one obtains $(4K + \mu^2)W_{\text{tot}} = \tilde{\mu}^2$, and using Eq. (70) gives $W_{\text{tot}}^{(K=-1,0)} = 1$.

Our answer to Mach's question "what share?" is given by (1) the *radial weight function* $W(\chi)$ with its Yukawa-force cutoff, and (2) the *angular weight function* $\vec{X}_{\ell=1,m}^{-*}$ of Eq. (56) in the projection of the general velocity field on the toroidal vorticity sector with $\ell = 1$.

The *kinetic angular momentum* \vec{L} per $d\chi$ is directly obtained from $\vec{\Omega}_{\text{matter}}$ using the moment of inertia per $d\chi$ of a fluid shell, which is $(8\pi/3)(\rho + p)R^4$,

$$\frac{d\vec{L}}{d\chi} = \frac{8\pi}{3}(\rho + p)R^4\vec{\Omega}_{\text{matter}}(\chi). \quad (71)$$

Evidently the *kinetic* angular momentum \vec{L} is determined by angular velocity measurements *without prior knowledge* of $g_{0\phi}$, which is the output of solving Einstein's equations.

The *gravito-magnetic moment* per $d\chi$, $d\vec{\mu}_g/d\chi$, is defined in the same way as in ordinary magnetism, except that the current of charge is replaced by the current of energy \vec{J}_ϵ :

$$\frac{d\vec{\mu}_g}{d\chi} = \frac{1}{2} \frac{d\vec{L}}{d\chi} = \frac{1}{2}(4\pi R^2) \int_0^\infty d\Omega [\vec{R} \times \vec{J}_\epsilon], \quad (72)$$

$$\int d\Omega [\vec{n} \times \vec{J}_\epsilon]_m = -\frac{4\pi}{3} \int d\Omega < \vec{X}_{\ell=1,m}^{-*} \cdot \vec{J}_\epsilon >, \quad (73)$$

where \vec{n} is the radial unit vector at the source point, and $\vec{R} = R\vec{n}$. The last two equations show that the gravitomagnetic moment and the angular momentum involve a projection of the energy-current field on the sector of toroidal vorticity fields with $\ell = 1$.

$\vec{\Omega}_{\text{gyro}}$ is determined by an integral over the density of kinetic angular momentum: From Eqs. (9, 71)

$$\vec{B}_g(P) = -2\vec{\Omega}_{\text{gyro}} = -4G_N \int d(\text{vol}_Q) \left\{ \frac{1}{R} \frac{d\vec{L}}{d(\text{vol})} \right\} Y_\mu(\chi). \quad (74)$$

Expressing the kinetic angular momentum \vec{L} by the energy current \vec{J}_ϵ we obtain the *fundamental law* for *cosmological gravitomagnetism* in *integral form* for an *open* FRW universe, ($K = -1, 0$):

$$\begin{aligned} \vec{B}_g(P) &= -2\vec{\Omega}_{\text{gyro}}(P) = \\ &= -4G_N \int d(\text{vol}_Q) [\vec{n}_{PQ} \times \vec{J}_\epsilon(Q)] Y_\mu(\chi_{PQ}). \end{aligned} \quad (75)$$

Again: $\vec{\Omega}_{\text{gyro}}$ is given by the sources \vec{J}_ϵ at the *same time*. This follows directly from: (1) The facts that dragging effects are given directly by the momentum constraint, and that the momentum constraint in the vorticity sector, Eq. (5), for the general *time-dependent* context of

gravitomagnetodynamics is an *elliptic equation* (it has no partial time-derivatives). These facts have already been recognized by [11, 13, 14] among others. — (2) The fact that the tensor sector (gravitational waves) in linear perturbation theory (and of course the scalar sector) cannot produce a gravitomagnetic field \vec{B}_g nor a torque on a gyroscope has been demonstrated in Sec. V A.

The *vector structure* of Eqs. (72, 73, 75) needs explanation, since there is no global parallelism on a hyperbolic 3-space H^3 . This is explained at the end of Sec. I G.

The differences between Eq. (75) and *Ampère's law* in integral form are:

1. the replacement of the current of electric charge J_q by the energy current J_ϵ ,
2. the factor $(-G_N)$, as in the transition from Coulomb's law to Newton's law,
3. the additional factor 4, which occurs in the transition from Ampère's law of ordinary magnetism to gravitomagnetism,
4. the replacement of the $1/r^2$ force in Ampère's law by the *Yukawa force* $Y_\mu(\chi)$ for *cosmological* gravitomagnetism,
5. the need to distinguish between R (in the denominator of the Yukawa force) and χ (in the exponent and in the derivative) for FRW backgrounds with *curved* 3-space, $K = \pm 1$,
6. the remarkable fact that the law (75) for cosmological gravitomagnetism is valid for general *non-stationary* situations, while Ampère's law is valid only for stationary currents.

What *distance interval* χ of sources *dominates dragging* of local inertial frames at P ? — For a spatially flat universe, the weight function $W(\chi)$ grows linearly from $\chi = 0$ to $\chi \approx \mu^{-1} = \tilde{\mu}^{-1}$, and it decays exponentially for $\chi \gg \mu^{-1}$. Hence the sources Q around $\chi_{PQ} = \mu^{-1}$ dominate the frame-dragging at P . — For a spatially hyperbolic universe, $K = -1$, and for $\tilde{\mu} \gg a_c^{-1} \equiv 1$ the situation is approximately the same as for $K = 0$, since the frame dragging is caused by sources at distances much smaller than the curvature radius. — But for $K = -1$ and for χ much larger than the curvature radius $a_c \equiv 1$, the effective exponential cutoff factor in the weight function $W(\chi)$ is $\exp[-(\mu - 2)\chi]$. From $(\rho + p) > 0$ and from Eqs. (69, 70) with $K = -1$, it follows that $\tilde{\mu} > 0$ and $\mu > 2$. For $\tilde{\mu} \ll 1$ we have $(\mu - 2) \approx \tilde{\mu}^2/4$, the cutoff factor becomes $\exp(-\tilde{\mu}^2\chi/4)$, and we have three regimes for $W(\chi)$: For $0 \leq \chi \ll 1$ the weight function $W(\chi)$ grows linearly, for $1 \ll \chi \ll \tilde{\mu}^{-2}$ the weight function is constant, and for $\chi \gg \tilde{\mu}^{-2}$ it decays exponentially. On a logarithmic scale the sources near $\chi = \tilde{\mu}^{-2}$ dominate frame dragging for $\tilde{\mu} \ll 1$. — For all open universes the dragging of inertial frames is dominated by sources

at a distance equal or larger than the $(\rho + p)$ -radius: Local inertial frames are rigidly “in the grip of the distant universe”.

Form-invariance of the solution Eq. (75) under transformation to a rigidly rotating frame: If the reference FIDO at the position of the gyroscope changes his local spatial axes to new ones, \vec{e}_i^* , with angular velocity $-\vec{\Omega}^*(0)$ relative to the old ones, Eq. (75) remains valid as it stands, because both sides of Eq. (75) change by the same term. On the right-hand-side $(\vec{n} \times \vec{J}_\varepsilon)$ changes by $[-(\rho + p)\Omega^*(0)R \sin \theta \vec{e}_\theta]$, where we have put the z -axis of the spherical coordinates in the direction of $-\vec{\Omega}^*$. The integration over the solid angle gives $(8\pi/3)(\rho + p)R\vec{\Omega}^*(0)$. The integration over χ is the same as in Eq. (11). The resulting change on the right-hand side is $[-2\vec{\Omega}^*(0)]$, which is equal to the change on the left-hand side.

To obtain definite values for measurements of fields \vec{J}_ε and \vec{B}_g , one must make an arbitrary choice of a definite state of rotation of FIDO-axis directions along *one* world line, as explained at the end of Sec. IB. But for *any* such choice: The field of *measured energy currents* \vec{J}_ε all by itself *completely determines* the gravitomagnetic field \vec{B}_g at all points, i.e. the *precession* $\vec{\Omega}_{\text{gyro}}$ of *gyroscopes* at all points: *No absolute element* is needed in the input, as required in the starting point of Mach [4, 5].

For cosmological gravitomagnetism and for *given sources* \vec{J}_ε , measured via angular velocities, distances, and $(\rho + p)$: The solution Eq. (75) for the momentum constraint is *unique*. There exists no ambiguity of adding a solution of the homogeneous equation. *No regular solution* exists for the *homogeneous* momentum constraint in cosmological gravitomagnetism, $(\Delta - \mu^2)\vec{A}_g = 0$. — Therefore for regular solutions *no boundary conditions* are needed at spatial infinity for open universes. See also Sect. XI of [7].

D. Exact dragging of inertial axes for perturbations of a closed FRW universe

To find the Green function for \vec{A}_g in a closed universe and in the toroidal sector, we note that a toroidal vector field $\vec{V}_{\ell=1,m}^{\text{tor}}$ which is a *regular vector field* at $\chi = 0$ and at $\chi = \pi$ must have a point-wise norm which *vanishes* linearly at these two points. Therefore the radial solution $G_A^{\text{tor}} = f_A^{\text{tor}}$, which multiplies $\vec{X}_{\ell=1,m}^-$, must also vanish linearly at $\chi = 0$ and at $\chi = \pi$. Hence the radial part of the Green function can be written as

$$f_A^{\text{tor}}(\chi, \chi_s) = \bar{N} \tilde{j}(\chi_{<}) \tilde{j}(\pi - \chi_{>}). \quad (76)$$

We have again dropped the subscripts ($q = i\mu, \ell = 1$) and the superscript ($K = +1$) for better readability.

The *discontinuity* in the derivative of the radial part of the Green function, the analog of Eq. (63), is

$$\text{disc} \left[\frac{d}{d\chi} f_A^{\text{tor}} \right]$$

$$= \bar{N} W \left[\tilde{j}(\chi), \tilde{j}(\pi - \chi) \right]_s. \quad (77)$$

To evaluate the Wronskian W we use the method given before Eq. (64). Specifically, $(R^2 W)$ is independent of χ , and we evaluate this expression for $\chi \rightarrow 0$. We use $\tilde{j}_{q=i\mu, \ell=1}^{(K=+1)}(\chi) \rightarrow (i/3)\chi\mu(1 + 1/\mu^2)$ and $\tilde{j}_{q=i\mu, \ell=1}^{(K=+1)}(\pi - \chi) \rightarrow i/(\chi^2\mu^2) \sinh(\mu\pi)$ for $\chi \ll 1$, and we obtain

$$\begin{aligned} W[\tilde{j}_{q,\ell}^{(K)}(\chi), \tilde{j}_{q,\ell}^{(K)}(\pi - \chi)]_{q=i\mu, \ell=1}^{K=+1} \\ = \frac{1}{\mu R^2} \left(1 + \frac{1}{\mu^2}\right) \sinh(\mu\pi). \end{aligned} \quad (78)$$

Hence the normalization \bar{N} is

$$\bar{N} = N \sinh^{-1}(\mu\pi), \quad (79)$$

where $N \equiv N^{(K=0,-1)}$ is given by Eq. (66).

The gravitomagnetic field \vec{B} at the origin generated by a rotating shell $\Omega(\chi) = \delta(\chi - \chi_s)\Omega_s$ on a FRW background with $K = +1$ is given by taking the result (67) for $K = (0, -1)$ and making the replacement

$$\exp(-\mu\chi) \Rightarrow \sinh^{-1}(\mu\pi) \sinh[\mu(\pi - \chi)]. \quad (80)$$

For an arbitrary velocity field it directly follows with this replacement from the result of Eq. (9) that $\vec{\Omega}_{\text{gyro}}$ is equal to the weighted average of $\vec{\Omega}_{\text{matter}}$,

$$\begin{aligned} \vec{\Omega}_{\text{gyro}} &= \int_0^\pi d\chi \vec{\Omega}_{\text{matter}}(\chi) W(\chi), \\ W(\chi) &= \frac{1}{3} \tilde{\mu}^2 R^3 \sinh^{-1}(\mu\pi) \\ &\quad \frac{-d}{d\chi} \left\{ \frac{1}{R} \sinh[\mu(\pi - \chi)] \right\}. \end{aligned} \quad (81)$$

Again, the averaging weight function is normalized to unity, $W_{\text{tot}} = \int_0^\pi d\chi W(\chi) = 1$. The proof uses the same steps as the ones given to prove unit normalization of $W(\chi)$ for open universes in the second paragraph of Sec. VC. Hence for perturbations of a closed FRW universe the evolution of inertial axes *exactly follows* the *weighted average* of cosmic matter motion, there is *exact dragging* of inertial axes by cosmic energy currents, as stated in Mach's postulate.

The fundamental law for gravitomagnetism in integrated form follows with the replacement (80) from Eq. (75),

$$\begin{aligned} \vec{B}(P) &= -4G_N \sinh^{-1}(\mu\pi) \int d(\text{vol}_Q) (\vec{n}_{PQ} \times \vec{J}_\varepsilon) \\ &\quad \frac{-d}{d\chi} \left\{ \frac{1}{R} [\sinh[\mu(\pi - \chi)]] \right\}. \end{aligned} \quad (82)$$

The solutions (81, 82) are again *frame-invariant*, i.e. form-invariant when going to a reference frame which is in globally rigid rotation relative to the previous reference frame. The proof is identical to the one given for open universes in Sec. VC.

Einstein, Wheeler, and others, recently e.g. Bičák et al [16], have held the view that from the Machian point of view closed universes are preferable. In contrast our results show that Mach's hypothesis, exact dragging of inertial axes, holds for linear perturbations of *all* FRW universes, open or closed.

E. Exact dragging of inertial axes for perturbations of Einstein's static, closed universe

For a static, closed universe $\mu^2 \equiv -4(dH/dt) = 0$, and $(\tilde{\mu}/2)^2 = K = +1$. Taking the limit $\mu \rightarrow 0$ of the result for a FRW universe with $K = +1$, Eq. (82), one obtains the fundamental law for gravitomagnetism for Einstein's static, closed universe,

$$\vec{B}(P) = -4G_N \int d(\text{vol}_Q) (\vec{n}_{PQ} \times \vec{J}_\epsilon) \frac{-d}{d\chi} \left[\frac{1}{R} (1 - \chi/\pi) \right] \quad (83)$$

Again $\vec{\Omega}_{\text{gyro}}$ is equal to the weighted average of $\vec{\Omega}_{\text{matter}}$,

$$\begin{aligned} \vec{\Omega}_{\text{gyro}} &= \int_0^\pi d\chi \vec{\Omega}_{\text{matter}}(\chi) W(\chi), \\ W(\chi) &= \frac{4}{3} R^3 \frac{-d}{d\chi} \left[\frac{1}{R} (1 - \chi/\pi) \right], \end{aligned} \quad (84)$$

where $W_{\text{tot}} = \int_0^\pi d\chi W(\chi) = 1$. Eq. (84) states that also for perturbations of Einstein's static closed FRW universe the evolution of inertial axes *exactly follows* the weighted average of cosmic matter motion, there is *exact dragging* of inertial axes by cosmic energy currents, as stated in Mach's postulate. — The solution (83) is again *form-invariant* when going to a reference frame which is in globally rigid rotation relative to the previous reference frame.

Einstein's static, closed universe (3-sphere) has been important in the work of Ozsváth and Schücking [53], who added a Bianchi IX amplitude (the lowest mode of tensor perturbations) to Einstein's universe, and then investigated Mach's principle. Unfortunately Ozsváth and Schücking considered the vanishing of the *local* vorticity relative to a gyroscope to be a test of Mach's principle. This is in contradiction to our conclusion that the *weighted cosmic average* of the energy currents in Eq. (84) directly determines the time-evolution of gyroscope axes. Our result agrees with Mach's hypothesis that some (to him unknown) cosmic average of matter motion would determine the time-evolution of gyroscope axes.

F. Mach's principle for the limits to the Milne and the de Sitter universe

Many physicists (but definitely not Mach) have asked again and again: "If you could remove a body far away from all sources of gravity, what would be the inertial

behaviour of this body?" — If the density of galaxies was e.g. 10^{-9} per Hubble volume in an otherwise empty universe, a particle could be a distance of 10^3 Hubble radii away from all other matter. All the same, if galaxies in the unperturbed universe were homogeneously and isotropically distributed, we have shown that for a universe with linear vorticity perturbations *Mach's principle still holds*.

The *Milne universe* is the limit $\rho/\rho_{\text{crit}} \rightarrow 0$ (for p/ρ fixed) of a FRW universe. The first Friedmann equation gives $H^2 + K/a_c^2 \rightarrow 0$, hence we have a hyperbolic universe with a curvature radius a_c equal to the Hubble radius H^{-1} . The Milne universe is equivalent to a part of Minkowski space-time, the forward light-cone originating at the space-time point where the Hubble expansion (of test particles at rest in the Milne universe) started.

How can exact dragging of inertial axes by cosmic energy flows work for $\rho/\rho_{\text{crit}} \ll 1$, when the total energy within a Hubble volume is arbitrarily small? According to the discussion in Sec. V C, the weight function becomes $W(\chi) \approx (\tilde{\mu}^2/4) \exp(-\tilde{\mu}^2\chi/4)$ for $\tilde{\mu}^2 \equiv 16\pi G_N(\rho + p) \ll 1$. The *prefactor* $(\tilde{\mu}^2/4)$ is *arbitrarily close to zero*, but the integral for W_{tot} would be linearly divergent without the cutoff at $\chi_{\text{cutoff}} = 4\tilde{\mu}^{-2}$. Therefore the *integral without the prefactor* gives $(4/\tilde{\mu}^2)$, and this is *arbitrarily large*. The product of these two factors gives $W_{\text{tot}} = 1$, exact dragging of inertial axes, as demonstrated in Sec. V C.

The *de Sitter universe* is the limit $(p/\rho) \searrow (-1)$ of FRW universes with $K = 0, \pm 1$. These three cases correspond to three different slicings of one de Sitter space-time geometry or part of it.

How does Mach's principle work arbitrarily close to the de Sitter limit, i.e. for $[(p/\rho) + 1] \ll 1$ and ρ/ρ_{crit} finite and fixed, when the energy currents within the Hubble volume can be made arbitrarily small for any observer?

For $K = 0$ we have $\mu^2 = \tilde{\mu}^2 = 16\pi G(\rho + p)$. Hence the prefactor in $W(\chi)$ is of order $\mu^2 \ll 1$. On the other hand the integral for W_{tot} would be quadratically divergent without the cutoff at $\chi_{\text{cutoff}} = \mu^{-1}$. Hence the integral without the prefactor is of order $1/\mu^2 \gg 1$. The two effects cancel, and $W_{\text{tot}} = 1$ as stated in Eq. (11).

For $K = -1$ the discussion of the de Sitter limit, $(\rho + p) \rightarrow 0$, is exactly the same as the discussion given above for the Milne limit.

For $K = +1$ and $(\rho + p) \ll \rho$, i.e. $\tilde{\mu} \ll 1$, one obtains $\mu \rightarrow \pm 2i(1 - \tilde{\mu}^2/8)$, and $W(\chi) = 8(3\pi)^{-1} \sin^4 \chi$, i.e. $\tilde{\mu}^2$ drops out. This gives $\int_0^\pi W(\chi) d\chi = 1$.

G. Globally rigid rotations: Zero modes of $(\Delta + 4K)$

We have seen in Sec. I B that for vorticity perturbations the fixed-time slices Σ_t are uniquely determined, and that each slice has the unperturbed 3-geometry of $\{E^3, S^3, H^3\}$. For the unperturbed 3-geometries we have chosen unperturbed 3-coordinates: spherical coordinates for $\{E^3, S^3, H^3\}$, resp. Cartesian coordinates for E^3 . This does not yet fix the coordinatization (gauge) com-

pletely, because there still exist *residual gauge transformations* by *time-dependent globally rigid rotations* of the spherical coordinates for 3-space. In this subsection we show that these residual gauge transformations are *zero modes* of $(\Delta + 4K)$, where $a_c = 1$.

Since this paper on Mach's principle focusses on rotational motion of fiducial axes, not on linear acceleration of frames, we apply a change of the spatially global (rigid) reference frame such that the 3-coordinate velocity dx^i/dt of the reference FIDO at the spatial origin, $\chi = 0$, is left unchanged. The angular velocity of the *new* reference-FIDO axes at the spatial origin *relative to the old* ones is denoted by $[-\vec{\Omega}^*(\chi = 0)]$.

In the new coordinates the matter 3-velocity field is changed by the additional field $\vec{v}_{\text{matter}}^*$, which is equal to the negative of the additional shift vector field $\vec{\beta}^*$ and also equal to the negative of the additional gravitomagnetic vector potential \vec{A}_g^* ,

$$\vec{v}_{\text{matter}}^* = -\vec{A}_g^* = \Omega^*(0) \vec{e}_\phi. \quad (85)$$

The momentum constraint, Eq. (5), for the additional fields $\vec{v}_{\text{matter}}^* = -\vec{A}_g^*$, with the definition of $\tilde{\mu}^2$ for the coefficient in the source term, Eq. (68), and with the relation $(\tilde{\mu}^2 - \mu^2) = 4K$, Eq. (70), gives

$$(\Delta + 4K)\vec{A}_g^* = 0. \quad (86)$$

i.e. the additional field \vec{A}_g^* arising from a globally rigid rotation is a *zero mode* of the operator $(\Delta + 4K)$.

The curl of \vec{A}_g^* gives the additional gravitomagnetic field \vec{B}_g^* ,

$$\begin{aligned} B_\chi^* &= -2\Omega^*(0) \cos \theta \\ B_\theta^* &= +2\Omega^*(0) \sin \theta \frac{dR}{d\chi} \\ B_\phi^* &= 0. \end{aligned} \quad (87)$$

For $K = 0$ (Euclidean 3-space) the additional \vec{B}_g^* -field and hence the $\vec{\Omega}_{\text{gyro}}^*$ -field are homogeneous fields. But for $K = \pm 1$ the additional \vec{B}_g^* -field is not a homogeneous field, its magnitude varies as a function of χ .

The additional fields $\vec{v}_{\text{matter}}^*$, \vec{A}_g^* , \vec{B}_g^* are equal to the fields, which arise in an *unperturbed FRW universe*, if the coordinate system is in rigid rotation relative to FRW coordinates, and if the FIDOs and their LONBs are adapted to the rotating coordinates. — But note that the constraint for a given *vanishing source field* $\vec{J}_e \equiv 0$ has the unique solution $\vec{A}_g \equiv 0$.

H. Six fundamental tests for the principle formulated by Mach

The following six tests cover different aspects, intimately related, of the principle formulated by Mach:

1. *Exact dragging* of local inertial axes by a weighted average of cosmological energy currents. The *weight function* must be *normalized to unity*.
2. *“Totally determined”*: The time-evolution of local inertial axes (relative to any given set of axes) is totally determined by cosmological energy currents.
3. *No absolute element as an input*. The time-evolution of local inertial axes must be an *output* determined by cosmic energy currents, *not an input*.
4. *“Solutions Frame-Invariant”*: The *solution* for $\vec{\Omega}_{\text{gyro}}$ in terms of cosmological energy currents must be *form-invariant* under transformations to globally rotating frames, i.e. *frame-invariant*.
5. *No boundary conditions* at spatial infinity in open cosmologies are needed to totally determine local non-rotating frames.
6. *No reference-frame condition* along one world line is needed to totally determine local non-rotating frames: In open universes such a reference-frame condition along a world line at spatial infinity is equivalent to boundary conditions. In closed universes such reference-frame conditions along one world line play the analogous role to boundary conditions for open universes.

Tests (1)-(4) represent different aspects of the postulate as formulated by Mach. — Test (3) is the postulate of the “relativists” (Huyghens, Bishop Berkeley, Leibniz, Mach, and others) that only *relative motion* of physical objects is relevant. This was the *starting point* of Mach. — Test (4) is a necessary condition, formulated by Mach in his earliest writings [5] quoted in Sec. I A. — Test (1) is an *explicit formulation of Mach's Principle*, given in Sec. I A, while Tests (2)-(6) are less explicit about the implementation. — Tests (1) and (2) are unambiguous for *linear* perturbation theory. In the *nonlinear* case, there is now agreement that gravitational waves make contributions. Hence Einstein's formulation of Mach's principle [54], “the (metric tensor) field $g_{\mu\nu}$ is entirely determined ... by the energy tensor of matter” is untenable. But the definition of “energy currents” (= “momentum densities”) and “angular momentum densities” of the gravitational field has been controversial. Frame-dragging effects by “the angular momentum of gravitational waves” has been investigated in special, non-cosmological models in [55, 56]. — Tests (3) and (4) are meaningful in the presence of gravitational waves, even if the concept of energy currents of the gravitational field is not available in a precise way. — Tests (5) and (6) are intimately related.

The *kinetic energy currents*, i.e. the *LONB components* $T_{\hat{k}}^0$ can be *measured without prior knowledge* of the local non-rotating frames. Hence *no absolute element* is needed in the input for our approach. — In contrast the

canonical energy currents, i.e. the coordinate-basis components T_k^0 cannot be measured without the prior knowledge of the local non-rotating frames all over the universe. Hence using coordinate-basis components T_k^0 means that one needs an absolute element as an input, e.g. in the approach emphasized and advocated by [13, 16, 19].

The six tests listed above are *not fulfilled explicitly* (manifestly), if one uses the momentum constraint for the coordinate-basis components T_k^0 , as done and advocated e.g. in [13, 16, 19]. These papers give no proof that any one of the six tests is satisfied.

Einstein's theory of General Relativity applied to the *solar system* fails tests (1)-(5). Hence it is not a "theory of relativity" in the sense of the relativists before Einstein, as has been recognized for a long time.

In this paper we have shown that the six fundamental tests listed above are *manifestly (explicitly) fulfilled* for linear perturbations of FRW universes in *Cosmological General Relativity*.

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APPENDIX: THE DE RHAM-HODGE LAPLACIAN FOR VECTOR FIELDS IN RIEMANNIAN 3-SPACES: CONNECTING THE CALCULUS OF DIFFERENTIAL FORMS AND THE CURL NOTATION

The material in this appendix (in addition to Sec. IE) is essential for cosmological gravitomagnetism, because:

(1) For vector perturbations the *de Rham-Hodge Laplacian* ("the" Laplacian in the mathematics literature) has many crucial advantages (listed below) over ∇^2 , the "rough Laplacian", which, unfortunately, has been used in the entire literature on cosmology. The calculus of *differential forms* is needed for the de Rham-Hodge Laplacian. In addition, this calculus makes *computations simple*, since no Christoffel symbols are required for computations. Unfortunately this calculus has not been used in the literature on cosmological vector perturbations.

(2) On the other hand the *elementary notation* for vector calculus in Riemannian 3-spaces with *curl* and *div* is indispensable for *physical insight* in gravitomagnetism, i.e. for seeing directly that the mathematical structure of the momentum constraint for cosmological gravitomagnetism on FRW backgrounds with $K = (\pm 1, 0)$ is *explicitly form-identical* (without curvature terms) to the familiar structure of Ampère magnetism in Euclidean 3-space (except for the Yukawa cutoff at the H -dot radius).

(3) Unfortunately almost all books on differential geometry and on General Relativity, and all research papers

on cosmological perturbation theory do not treat the topics of the *de Rham-Hodge Laplacian* and the *Weitzenböck* formula. The latter gives the difference $(\Delta - \nabla^2)\vec{A}$, which is needed when comparing with the literature on cosmological vector perturbations. In books about differential geometry these topics are treated in [23, 24]. — But, as far as we know, no book on General Relativity or differential geometry and no paper on cosmological perturbations gives the *explicit equations* which connect the notation of differential forms with the elementary notation for vector calculus in Riemannian 3-spaces: $(\text{curl } \vec{a})_\mu = (\star d \vec{a})_\mu$, Eq. (A.2), and for vorticity fields $(\Delta \vec{a})_\mu = -(\text{curl curl } \vec{a})_\mu = -(\star d \star d \vec{a})_\mu$, Eqs. (A.10, and A.11). Ref. [23] does not give such equations, and Ref. [58] instead gives a "rough, symbolic identification", which cannot be used literally in the equation $\Delta \vec{a} = -\text{curl curl } \vec{a}$, see the paragraph following the identity (A.2).

Our aim in this Appendix is to give the derivations of the tools and results needed in cosmological gravitomagnetism. We present this for cosmologists with little or no experience in the calculus of differential forms.

The identities for vector calculus in Riemann 3-spaces at the level of *first* covariant derivatives ∇ are indistinguishable from identities for vector calculus in Euclidean 3-space, given e.g. in [57], because *curvature effects cannot appear* at the level of first derivatives. — At the level of *second* covariant derivatives ∇ , the standard identities of vector calculus in Euclidean 3-space, e.g. in [57], remain true in Riemannian 3-space, if and only if one uses the de Rham-Hodge Laplacian Δ as opposed to the 'rough Laplacian' ∇^2 .

The reasons which *single out* the de Rham-Hodge Laplacian in the vector sector on curved 3-space, and which give its *conceptual motivation* in physics:

1. If for a vector field all types of *sources* (div and curl) are *zero*, then the de Rham-Hodge Laplacian of this vector field is also zero, i.e. the vector field is harmonic; Sec. IE.
2. The de Rham-Hodge Laplacian *commutes* with curl, div, grad; Sec. IE.
3. The *identities* of vector calculus in Euclidean 3-space [57] and the important first identity of Green generalized to vorticity fields, Eq. (A.9), remain true in Riemannian 3-spaces, if and only if one uses the de Rham-Hodge Laplacian.
4. For electromagnetism in curved space-time the *equivalence principle forbids curvature terms*, if and only if the de Rham-Hodge Laplacian resp d'Alembertian is used; Sec. IE.
5. The *action principle* for Ampère magnetism in Riemannian 3-space, Eq. (A.6), which is *bilinear in first derivatives*, produces the Ampère equation with the de Rham-Hodge Laplacian and without curvature terms, Eq. (A.10).

6. The momentum constraint for cosmological *gravitomagnetism* has *identical mathematical structure* to *Ampère magnetism* (except for the Yukawa cut-off), if and only if the de Rham-Hodge Laplacian is used; Eq. (5).
7. The *Hodge decomposition* for closed Riemannian 3-spaces, specialized to 1-forms, states that any vector field can be uniquely decomposed into a sum of a gradient field plus a curl field plus a harmonic field. This theorem is valid, if and only if the de Rham-Hodge Laplacian is used to define harmonic fields.
8. *No Christoffel symbols* resp connection coefficients are needed to compute the de Rham-Hodge Laplacian; Eqs. (A.2, A.5, A.14, A.15).

Every one of these properties does *not* hold for the “rough Laplacian” ∇^2 .

1. Differential forms and curl, grad, div

One can *represent* a vector field \vec{v} in Riemannian 3-space, a geometric object, by its Local Ortho-Normal Basis (LONB) components, denoted by a *hat* over the index, $v_{\hat{i}}$, or by its contravariant components in a coordinate basis v^λ , or by its associated 1-form \tilde{v} , resp. by its 1-form components (covariant components) v_λ . The calculus of differential forms is fundamentally tied to the representation in some unspecified *coordinate basis* and with *covariant* components.

For the *vector product* $(\vec{a} \times \vec{b})$ the two notations are connected by the equation

$$(\vec{a} \times \vec{b})_\lambda = g_{\lambda\kappa} \varepsilon^{\kappa\mu\nu} a_\mu b_\nu = (\star[\tilde{a} \wedge \tilde{b}])_\lambda, \quad (\text{A.1})$$

where \tilde{a} and \tilde{b} are the 1-forms (covariant vectors) associated with \vec{a} and \vec{b} , i.e. $(\tilde{a})_\mu = (\vec{a})_\mu = a_\mu$. A tilde $\tilde{}$ designates p -forms (totally antisymmetric covariant tensors of rank p). — The first step is the *wedge product* \wedge (exterior product, antisymmetric product) of two 1-forms, which gives a 2-form \tilde{c} with components $c_{\mu\nu} = [\tilde{a} \wedge \tilde{b}]_{\mu\nu} \equiv a_\mu b_\nu - a_\nu b_\mu$. — The second step is taking the *Hodge dual* (Hodge star operator) of the resulting 2-form. All our bases will have right-handed orientation. The primary definition for the Hodge dual is given in any LONB: One contracts with the totally antisymmetric ε -tensor, the *Levi-Civita tensor*, in a LONB, where $\varepsilon_{\hat{1}\hat{2}\hat{3}} \equiv +1$. Explicitly: the Hodge dual of a p -form \tilde{c} is given by $(\star\tilde{c})_{\hat{i}_1, \dots, \hat{i}_{n-p}} = (1/p!) \varepsilon_{\hat{i}_1, \dots, \hat{i}_{n-p}, \hat{j}_1, \dots, \hat{j}_p} c_{\hat{j}_1, \dots, \hat{j}_p}$, where $n = 3$ is the dimension of our Riemann space. Example: $\tilde{h}^{(1)} = \star\tilde{b}^{(2)}$ written in LONB components gives $h_{\hat{1}} = b_{\hat{2}\hat{3}}$. It follows that $\star\star = 1$ for p -forms in a Riemannian 3-space. — The *volume element* spanned by a LONB with postive orientation is equal to $+1$: $v(\vec{e}_1, \vec{e}_2, \vec{e}_3) = \varepsilon(\vec{e}_1, \vec{e}_2, \vec{e}_3) = \varepsilon_{\hat{1}\hat{2}\hat{3}} = +1$. The LONB components of the ε -tensor are *invariant under proper rotations* (as are the

LONB components of the metric tensor, $g_{\hat{i}\hat{j}} = \delta_{ij}$). — But in the calculus of differential forms the Hodge dual involves the totally antisymmetric ε -tensor in a *coordinate basis*, $\varepsilon_{\lambda\mu\nu} = \varepsilon_{\hat{\lambda}\hat{\mu}\hat{\nu}} \sqrt{g}$ and $\varepsilon^{\lambda\mu\nu} = \varepsilon_{\hat{\lambda}\hat{\mu}\hat{\nu}} (\sqrt{g})^{-1}$. The volume of a parallelepiped spanned by a triple of vectors is $v(X, Y, Z) = \varepsilon_{\lambda\mu\nu} X^\lambda Y^\mu Z^\nu$, hence the volume 3-form \tilde{v} has components $v_{\lambda\mu\nu} = \varepsilon_{\lambda\mu\nu}$. — For the Hodge dual which takes a p -form into a $(3-p)$ -form, one can either first raise the indices of the original p -form and then act with $\varepsilon_{\lambda\mu\nu}$, or equivalently one can first act with $\varepsilon^{\lambda\mu\nu}$ and then lower the indices to obtain the final $(3-p)$ -form.

The operation *curl* takes a vector field \vec{a} into another vector field of opposite parity, $(\text{curl } \vec{a}) = (\vec{\nabla} \times \vec{a})$, where ∇ is the covariant derivative. The formula analogous to Eq. (A.1) gives,

$$(\text{curl } \vec{a})_\lambda = (\vec{\nabla} \times \vec{a})_\lambda = g_{\lambda\kappa} \varepsilon^{\kappa\mu\nu} \partial_\mu a_\nu = (\star d\tilde{a})_\lambda. \quad (\text{A.2})$$

The last and the next-to-last expressions in Equations (A.2) give the *simplest method to compute* a curl. But the first and second expressions are crucial to make the *precise connection* with the notation universally used in electrodynamics [43] via an *identity* in the form of an *explicit equation*. — The first step on the right-hand side of Eq. (A.2), the *exterior derivative* (antisymmetric derivative) d of a 1-form \tilde{a} gives the 2-form $d\tilde{a}$ with components $(d\tilde{a})_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu$. Because the calculus of forms is tied to (unspecified) coordinate bases and to totally antisymmetric covariant tensors, the covariant derivative ∇_μ , after antisymmetrization, can be replaced by the ordinary partial derivative ∂_μ . This is the *first great simplification*, and it takes place, because the connection coefficients in the coordinate basis, the Christoffel symbols $\Gamma_{\mu\nu}^\rho$, are symmetric in the second and third indices. — Compared to other identities given for curl in the literature [24], the identity $(\text{curl } \vec{a})_\lambda \equiv (\star d\tilde{a})_\lambda$ has the important advantage that it directly transcribes the computational prescription of elementary vector calculus for Cartesian coordinates, $(\text{curl } \vec{a})_i = \varepsilon_{ijk} \partial_j a_k$, to Riemannian 3-spaces (or to curvilinear coordinates in Euclidean 3-space). — Some authors, e.g. [58], make a ‘dictionary’ with the ‘rough, symbolic identification’ $\text{curl } \mathbf{A} \Leftrightarrow d\alpha^1$, where the 1-form α^1 is the covariant expression for the vector \mathbf{A} . This identification cannot be used literally in the operation $(\text{curl } \text{curl } \mathbf{A})$, because the Hodge star operator is missing in front of $d\alpha^1$ in their rough, symbolic identification. The operation $(\text{curl } \text{curl})$ shows that the output of the operation curl cannot be the 2-form $d\alpha^1$, the output must be a vector (represented e.g. by a 1-form) as in our identity (A.2), which directly produces the identity $(\text{curl } \text{curl } \vec{a})_\lambda = (\star d\star d\tilde{a})_\lambda$.

The operation *grad* takes a scalar field ϕ into $\vec{\nabla}\phi$.

$$(\text{grad } \phi)_\lambda = (\vec{\nabla}\phi)_\lambda = \partial_\lambda \phi = (d\phi)_\lambda. \quad (\text{A.3})$$

The *inner product* (scalar product) of two vectors can be rewritten as an *antisymmetric product* by first going

to the dual of \tilde{b} ,

$$\vec{a} \cdot \vec{b} = a^\lambda b_\lambda = \frac{1}{2} \varepsilon_{\lambda\mu\nu} [a^\lambda (\varepsilon^{\mu\nu\kappa} b_\kappa)] = \star [\tilde{a} \wedge (\star \tilde{b})], \quad (\text{A.4})$$

where we have used $\frac{1}{2} \varepsilon_{\alpha\beta\gamma} \varepsilon^{\beta\gamma\delta} = \delta_\alpha^\delta$. The wedge product of a 1-form \tilde{a} with a 2-form \tilde{c} is given by $[\tilde{a} \wedge \tilde{c}]_{\lambda\mu\nu} = a_\lambda c_{\mu\nu} + a_\mu c_{\nu\lambda} + a_\nu c_{\lambda\mu} = [\tilde{c} \wedge \tilde{a}]_{\lambda\mu\nu}$. Similarly the inner product of two p -forms α, β is given by $\langle \alpha, \beta \rangle = \star(\alpha \wedge \star \beta)$.

The *divergence*, $\text{div } \vec{a} = \vec{\nabla} \cdot \vec{a}$, can be written as an *antisymmetric derivative* in analogy with Eq. (A.4). This is the *second great simplification*, because the antisymmetrization again eliminates the need for Christoffel symbols. It provides the easiest way to obtain the explicit expression for $(\vec{\nabla} \cdot \vec{a})$ at the end of the following sequence of equations,

$$\begin{aligned} \text{div } \vec{a} &= \vec{\nabla} \cdot \vec{a} = \star d \star \tilde{a} \\ &= \varepsilon^{\alpha\beta\gamma} \partial_\alpha (\varepsilon_{\beta\gamma\delta} a^\delta) = (1/\sqrt{g}) \partial_\mu (\sqrt{g} a^\mu). \end{aligned} \quad (\text{A.5})$$

Some well-known textbooks on General Relativity need more than a dozen steps to derive the well-known last expression in Eq. (A.5), because they work with Christoffel symbols. — But note that the expression $(\star d \star \tilde{a})$ is much more useful, since it gives e.g. $\text{div } \vec{x}^\pm$ of Eqs. (44, 46) in a trivial way.

2. The Laplacian on vector fields in curved 3-spaces

The Laplacian Δ acting on vector fields resp 1-forms in curved space is called de Rham Laplacian in [32], and [58] writes: “This Laplacian was defined first by Kodaira and independently by Bidal and de Rham.” The history including references is given in [59], footnote on p. 125. Since other authors use the name Hodge in this connection, and in order to be clearly understood, we shall call it the de Rham-Hodge Laplacian. The de Rham-Hodge Laplacian Δ must be distinguished from the “*rough Laplacian*” $\nabla^2 \equiv \nabla^\alpha \nabla_\alpha$ [25], [58]. Only when applied to a scalar ϕ field, one has $(\Delta - \nabla^2)\phi = 0$. — For the primary conceptual definition of the Laplacian acting on vector fields, we start from an *action integral*, because an action integral only involves first derivatives, hence it is independent of curvature effects. For *Ampère magnetostatics* in static curved 3-space, the action integral (times -1) reduces to the energy functional,

$$E[\vec{A}] = \int dv [(\text{curl } \vec{A})^2 - 4\pi \vec{A} \cdot \vec{J}], \quad (\text{A.6})$$

where $dv = d^3x \sqrt{g}$ is the invariant volume element. $E[\vec{A}]$ is given by the global scalar product of two vector fields, $(\dots) = \int dv \langle \dots, \dots \rangle$, where $\langle \dots, \dots \rangle$ is the point-wise scalar product.

To carry out the variation of the energy functional $E[\vec{A}]$ with respect to variations of \vec{A} , one has to perform

a partial integration. — To make the surface terms disappear, we must either have a compact Riemann space, or, in a non-compact Riemann space, one of the two forms, the variation, must have *compact support*. — For partial integration in curved space one needs the tool of differential forms. The *adjoint operator* to d in the sense of the global scalar product for compactly supported forms is denoted by d^* . Hence $(\beta^p, d\alpha^{p-1}) \equiv (d^* \beta^p, \alpha^{p-1})$, where the superscript p refers to p -forms. Expressing the point-wise inner product via a wedge product, Eq. (A.4), and using the explicit coordinate-basis expression, partial integration gives

$$d^* \alpha^p = (-1)^p \star d \star \alpha^p. \quad (\text{A.7})$$

The superscript $*$ denotes the adjoint, the non-superscripted \star denotes the Hodge dual. — The *codifferential operator* δ in this paper is defined with the opposite sign from the adjoint,

$$\delta \equiv -d^* \Rightarrow \delta \tilde{a} = \text{div } \vec{a}, \quad (\text{A.8})$$

where \tilde{a} is a 1-form. Some books define δ with the opposite sign from our sign, $\delta \equiv d^*$. — The exterior derivative d takes a p -form into a $(p+1)$ -form, the codifferential operator δ takes a p -form into a $(p-1)$ -form, and $dd = 0$, $\delta\delta = 0$.

Green's first identity generalized to vector fields with zero divergence on Riemannian 3-spaces (partial integration) reads

$$\int dv (\text{curl } \vec{\alpha}) \cdot (\text{curl } \vec{\beta}) = - \int dv \vec{\alpha} \cdot \Delta \vec{\beta} \quad (\text{A.9})$$

in the absence of surface terms. This identity is form-identical with the identity in Euclidean 3-spaces, and it can be used to give the *primary operational definition* of the *Laplace operator* (de Rham-Hodge Laplacian) on vector fields with zero divergence in Riemannian 3-spaces. — Equivalently, in our physics situation the primary concept of the Laplacian is given by starting from the energy functional Eq. (A.6): The factor multiplying the variation of \vec{A} in the integrand of Eq. (A.6) gives Ampère's law for \vec{A} , i.e. $\Delta \vec{A}$ in the *vorticity sector*,

$$\Delta \vec{A} = -\text{curl } \text{curl } \vec{A} = -4\pi \vec{J}_q \quad (\text{A.10})$$

$$(\Delta \vec{A})_\mu = -(\star d \star d \tilde{a})_\mu. \quad (\text{A.11})$$

Equation (A.11) with the operator $(\star d \star d)$ gives the simplest computational method in Riemannian 3-spaces (and for curvilinear coordinates in Euclidean 3-space). On the other hand, Eq. (A.10) with the operator $(\text{curl } \text{curl})$ in Riemannian 3-spaces is needed to make the precise connection using the notation universally adopted in electrodynamics in Minkowski space in a (3+1)-split [43].

In the general case of vector fields with $\text{div } \vec{a} \neq 0$ and $\text{curl } \vec{a} \neq 0$ the requirements for the Laplace operator Δ of de Rham and Hodge acting on vector fields are:

$$(1) \text{ 2nd order elliptic differential operator, self-adjoint,} \quad (\text{A.12})$$

$$(2) \text{ curl } \vec{a} = 0 \text{ and } \text{div } \vec{a} = 0 \Rightarrow \Delta \vec{a} = 0. \quad (\text{A.13})$$

Reversing the arrow in the statement (A.13) would be incorrect for *open* Riemannian 3-spaces (open universes) in contrast to statements in some textbooks. This is evident from simple counter-examples: For a FRW background with $K = 0$ a homogeneous \vec{B} -field gives $\Delta\vec{A} = 0$, but $\text{curl } \vec{A} \neq 0$. For a FRW background with $K = -1$ the analogous field is a poloidal vorticity field \vec{B} with $\ell = 1$ and $q = 0$, for which again $\Delta\vec{A} = 0$, but $\text{curl } \vec{A} \neq 0$. — From Eqs. (A.12, A.13) it follows that

$$\Delta\vec{a} = -\text{curl curl } \vec{a} + \text{grad div } \vec{a}. \quad (\text{A.14})$$

$$\begin{aligned} \Delta\tilde{a} &= (-\star d\star d + d\star d\star)\tilde{a} \\ &= (\delta d + d\delta)\tilde{a} \\ &= -(d + d^*)^2\tilde{a}. \end{aligned} \quad (\text{A.15})$$

Note that (dd^*) is self-adjoint, and $(d + d^*)^2 = (dd^* + d^*d)$ is self-adjoint and positive semi-definite.

The “*rough Laplacian*” $\nabla^2 = g^{\mu\nu}\nabla_\mu\nabla_\nu$ does *not* satisfy the requirement $\{\text{curl } \vec{a} = 0 \text{ and } \text{div } \vec{a} = 0\} \Rightarrow \Delta\vec{a} = 0$ in curved space, and it does *not* satisfy Green’s first identity Eq. (A.9).

3. The Weitzenböck Formula

The Weitzenböck formula [23, 60] for vector fields [24] and in 3-dimensional Riemannian space is de-

rived by first using $(\text{curl curl } A)_\alpha = -(\nabla^2 A)_\alpha + (\text{grad div } A)_\alpha + [(\nabla_\beta\nabla_\alpha - \nabla_\alpha\nabla_\beta)A]^\beta$. For the last term one uses the Ricci identity,

$$[(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)A]^\alpha = R^\alpha_{\beta\mu\nu}A^\beta, \quad (\text{A.16})$$

and the contracted Ricci identity,

$$[(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)A]^\mu = R_{\nu\beta}A^\beta, \quad (\text{A.17})$$

where $R^\alpha_{\beta\mu\nu}$ is the Riemann tensor and $R_{\nu\beta}$ the Ricci tensor. The sign conventions for the Riemann and Ricci tensors are those of Misner, Thorne, and Wheeler [20]. This gives the difference between the true Laplacian Δ (de Rham-Hodge Laplacian) and the “rough Laplacian” $\nabla^\alpha\nabla_\alpha$, i.e. the Weitzenböck formula,

$$(\Delta\tilde{A} - \nabla^2\tilde{A})_\alpha = R_\alpha{}^\beta A_\beta. \quad (\text{A.18})$$

For a FRW universe with $K = \pm 1$ and a curvature scale a_c , the Ricci tensor of 3-space is $R_\alpha{}^\beta = (2K/a_c^2)\delta_\alpha{}^\beta$, and the Weitzenböck formula gives

$$(\Delta - \nabla^2)\tilde{A} = -(2K/a_c^2)\tilde{A}. \quad (\text{A.19})$$

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- [1] H. Bondi and J. Samuel, Phys. Lett. A **228**, 121 (1997), gr-qc/9607009.
 - [2] <http://einstein.stanford.edu>
 - [3] I. Ciufolini and E.C. Pavlis, Nature **431**, 958 (2004).
 - [4] E. Mach, *Die Mechanik in Ihrer Entwicklung: Historisch-Kritisch Dargestellt* (Brockhaus, Leipzig, 1. Auflage 1883; 7., verbesserte und vermehrte Auflage 1912). English translation by T.J. McCormack, *The Science of Mechanics: A Critical and Historical Account of Its Development* (Open Court Publishing, La Salle, Ill., 6th American ed., 1960, based on the 9th German edition of 1933). See Chap. 2, Sec. 6, Subsec. 5.
 - [5] E. Mach, *Die Geschichte und die Wurzel des Satzes von der Erhaltung der Arbeit* (Calve, Prague, 1872; 2nd ed. Johann Ambrosius Barth, Leipzig, 1879); *History and Roots of the Principle of the Conservation of Energy* (Open Court, Chicago, 1911). See Note No. 1; pp. 77-80 are quoted in [8] on p. 107.
 - [6] C. Schmid, *Cosmological Vorticity Perturbations, Gravitomagnetism, and Mach’s Principle*, Proc. COSMO-01, Rovaniemi, Finland (2001), gr-qc/0201095
 - [7] C. Schmid, Phys. Rev. D **74**, 044031 (2006), gr-qc/0508066.
 - [8] J.B. Barbour and H. Pfister, editors, *Mach’s Principle: From Newton’s Bucket to Quantum Gravity* (Birkhäuser, Boston, 1995). Index of Different Formulations of Mach’s Principle on p. 530.
 - [9] H. Thirring, Phys. Zeitschr. **19**, 33 (1918), **22**, 29(E) (1921), English translation in Gen. Rel. Grav. **16**, 712 (1984).
 - [10] D.R. Brill and J.M. Cohen, Phys. Rev. **143**, 1011 (1966).
 - [11] L. Lindblom and D.R. Brill, Phys. Rev. D **10**, 3151 (1974).
 - [12] C. Klein, Class. Quantum Grav. **10**, 1619 (1993), and references therein.
 - [13] D. Lynden-Bell, J. Katz, and J. Bičák, Mon. Not. R. Astron. Soc. **272**, 150 (1995); **277**, 1600(E) (1995), and references therein.
 - [14] J. Katz, D. Lynden-Bell, and J. Bičák, Class. Quantum Grav. **15**, 3177 (1998).
 - [15] T. Doležal, J. Bičák, and N. Deruelle, Class. Quantum Grav. **17**, 2719 (2000).
 - [16] J. Bičák, D. Lynden-Bell, and J. Katz, Phys. Rev. D **69**, 064011 (2004).
 - [17] I. Ciufolini and J.A. Wheeler, *Gravitation and Inertia* (Princeton University Press, Princeton, NJ, 1995).
 - [18] H. Pfister, Gen. Relat. Gravit. **39**, 1735 (2007).
 - [19] J. Bičák, J. Katz, and D. Lynden-Bell, Phys. Rev. D **76**, 063501 (2007).
 - [20] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
 - [21] J.M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
 - [22] K.S. Thorne, R.H. Price, and D.A. MacDonald, editors, *Black Holes, The Membrane Paradigm* (Yale University Press, New Haven, 1986).
 - [23] J. Jost, *Riemannian Geometry and Geometric Analysis*, 4th ed. (Springer, Berlin, 2005), Secs. 2.1, 3.3.
 - [24] T. Frankel, *The Geometry of Physics*, 2nd ed. (Cam-

- bridge Univ. Press, Cambridge, 2004), Secs. 14.1, 14.2.
- [25] M. Berger, *A Panoramic View of Riemannian Geometry* (Springer, Berlin, 2003), Sec 15.6.
- [26] G. de Rham, *Differentiable Manifolds* (Springer, Berlin, 1984), Secs. 24 - 26. French original: *Variétés différentiables* (Hermann, Paris, 1955).
- [27] K. Tomita, Prog. Theor. Phys. **68**, 310 (1982).
- [28] H. Kodama and M. Sasaki, Progr. Theoret. Phys. Suppl. **78**, 1 (1984).
- [29] R. Durrer and N. Straumann, Helvetica Physica Acta, **61**, 1027 (1988).
- [30] N. Straumann, Ann. Phys. (Leipzig), **15**, 701 (2006), hep-ph/0505249.
- [31] W. Hu, U. Seljak, M. White, and M. Zaldarriaga, Phys. Rev. D **57**, 3290 (1998).
- [32] Misner, Thorne, and Wheeler [20], Sec. 22.4.
- [33] A. Lichnerowicz, Publ. Math. Inst. Hautes Et. Sci **10** (1961), 293 - 344, see p. 314.
- [34] A. Lichnerowicz, in *Relativity, Groups, and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964), see Eqs. (3.3, 3.4).
- [35] J. Ehlers in [8], p. 458.
- [36] A. Einstein, quoted by J. Ehlers in [8], p. 93.
- [37] Misner, Thorne, and Wheeler [20], Sec. 21.12, pp. 548-49.
- [38] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
- [39] Thorne et al [22], Secs. V.A.2 and V.A.3.
- [40] M. Bruni, P.K.S. Dunsby, and G.F.R. Ellis, Ap.J. **395**, 34 (1992).
- [41] E. Bertschinger and A.J.S. Hamilton, Ap.J. **435**, 1 (1994).
- [42] O. Heaviside, *A Gravitational and Electromagnetic Analogy*, Part I, The Electrician, **31**, 281-282 (1893), Part II, The Electrician, **31**, 359 (1893).
- [43] J.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 3rd ed., 1999, Sec. 9.7; 2nd ed., 1975, Sec. 16.2).
- [44] L.F. Abbott and R.K. Schaefer, Ap.J. **308**, 546 (1986).
- [45] Hu et al [31], Appendix B.
- [46] M.E. Rose, *Multipole Fields* (Wiley, New York, 1955).
- [47] A.R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton Univ. Press, Princeton, 1957).
- [48] T. Regge and J.A. Wheeler, Phys. Rev. **108**, 1063 (1957).
- [49] K.S. Thorne, Rev. Mod. Phys. **52**, 299 (1980).
- [50] E.T. Newman and R. Penrose, J. Math. Phys. **7**, 863 (1966).
- [51] J.N. Goldberg et al., J. Math. Phys. **8**, 2155 (1967).
- [52] C. Schmid, *The other half of Mach's principle: Frame dragging for linear acceleration*, unpublished.
- [53] I. Ozsváth, and E. Schücking, Nature (London) **193**, 1168 (1962), and Ann. Phys. (N.Y.) **55**, 166 (1969).
- [54] A. Einstein, Ann. Phys. (Leipzig) **360**, 241 (1918).
- [55] J. Bičák, J. Katz, and D. Lynden-Bell, Class. Quantum Grav. **25**, 165017 (2008).
- [56] D. Lynden-Bell, J. Bičák, and J. Katz, Class. Quantum Grav. **25**, 165018 (2008).
- [57] J.D. Jackson [43], vector formulas inside front cover (top half of page).
- [58] T. Frankel [24], p. 94.
- [59] G. de Rham, *Variétés Différentiables* (Hermann, Paris, 1955), see footnote on p. 125.
- [60] R. Weitzenböck, *Invariantentheorie* (Noordhoff, Groningen, 1923).